Some Results on Degenerate Elliptic Equations[1]

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Content

Chapter 1

Finitely Degenerate Elliptic Equations

1.1 Hypoellipticity and Sub-elliptic Estimate

1.1.1 Hörmander's Sum of Square Theorem

For $n \geq 2$, $\Omega \subset \mathbb{R}^n$ is an open domain, $X = \{X_1, X_2, \cdots, X_m\}$ is a system of real smooth vector fields defined on Ω. That is

$$X_j = \sum_{k=1}^{n} a_{jk}(x)\partial_{x_k}, \quad j = 1, \cdots, m,$$

where the real function $a_{jk}(x)$ belongs to $C^\infty(\Omega)$. If X and Y are real smooth vector fields, we can define the commutator:

$$[X, Y] = XY - YX. \tag{1.1.1}$$

Then it is easy to see that the commutator as a kind of product is linear respect to every variable, and also antisymmetric:

$$[X, Y] = -[Y, X].$$

Moreover, it holds the Jacobi identity: For three vector fields, it holds that

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0.$$

So, all real smooth vector fields not only constitute a vector space with respect to the real number field, but also form a Lie algebra in the view of the commutator. The Lie algebra induced by X (denoted by $\mathfrak{X}(X_1, X_2, \cdots, X_m)$) means the space spanned by

$$[X_{j_1}, [X_{j_2}, [X_{j_3}, \cdots [X_{j_{k-1}}, X_{j_k}] \cdots]]].$$

Also, the element of $\mathfrak{X}(X_1, X_2, \cdots, X_m)$ is a C^∞ real vector fields.

Definition 1.1.1 (Hörmander's condition). *For $n \geq 2$, the systems of real smooth vector fields $X = \{X_1, X_2, \cdots, X_m\}$ defined on an open domain Ω in $\mathbb{R}^n$. Let $J = (j_1, \cdots, j_k)$ with $1 \leq j_i \leq m$, we denote $|J| = k$. We say that $X = \{X_1, X_2, \cdots, X_m\}$ satisfies the*

Hörmander's condition on Ω if there exists a positive integer Q, such that for any $|J| = k \leq Q$, X together with all k-th repeated commutators

$$X_J = [X_{j_1}, [X_{j_2}, [X_{j_3}, \cdots [X_{j_{k-1}}, X_{j_k}] \cdots]]]$$

span the tangent space at each point of Ω. Here Q is called the Hörmander index of X on Ω, which is defined as the smallest positive integer for the Hörmander's condition above being satisfied.

Definition 1.1.2 (Finitely degenerate elliptic operator)**.** *If the real smooth system of vector fields X satisfies the Hörmander's condition on Ω with $1 < Q < +\infty$, then we say that X is a finitely degenerate system of vector fields on Ω and $\triangle_X = \sum_{i=1}^m X_i^2$ is a finitely degenerate elliptic operator on Ω.*

Example 1.1.1 (Kohn Laplacian operator)**.** *Let $X = (X_1, \cdots, X_N, Y_1, \cdots, Y_N)$, where*

$$X_j = \partial_{x_j} + 2y_j\partial_t, Y_j = \partial_{y_j} - 2x_j\partial_t, j = 1, \cdots, N.$$

defined on Heisernberg group $\Omega \subset \mathbb{R}^{2N+1}$, then the Kohn Laplacian $\triangle_X = \sum_{i=1}^N (X_i^2 + Y_i^2)$ is a finitely degenerate elliptic operator on Ω.

Example 1.1.2 (Grushin operator)**.** *Let $X = \{\partial_{x_1}, \cdots, \partial_{x_{n-1}}, x_1\partial_{x_n}\}$ defined on an open domain Ω of $\mathbb{R}^n$ which contains the origin, then $\triangle_X$ is a finitely degenerate elliptic operator on Ω.*

Let

$$L = \sum_{j=1}^m X_j^2(x) + X_0(x) + c(x),$$

where X_j, $j = 0, 1, \cdots, m$, are real smooth vector fields and $c(x)$ is a C^∞ function defined on Ω.

Definition 1.1.3 (Hypoellipticity)**.** *For all $u \in \mathcal{D}'(\Omega)$, if $Lu \in C^\infty(\Omega)$ implies $u \in C^\infty(\Omega)$. Then we say that the operator L is hypoelliptic on Ω.*

Theorem 1.1.1 (Hörmander's sum of square theorem, c.f.[24])**.** *If the real smooth system of vector fields X satisfies Hörmander's condition on Ω, then the operator L is hypoelliptic on Ω.*

For simplify, here we give a proof of Theorem 1.1.1 for the case $L = \sum_{j=1}^m X_j^2(x)$. First, we introduce the following pseudo-differential operator class.

Definition 1.1.4 (Symbol class $S^m(\Omega)$)**.** *Suppose Ω is an open set in $\mathbb{R}^n$ and m is a real number. The symbol class of order m on Ω, denoted by $S^m(\Omega)$, is the space of functions $p \in C^\infty(\Omega \times \mathbb{R}^n)$ such that for all multi-indices α and β and every compact set $K \subset \Omega$, there is a constant $C_{\alpha,\beta,K}$ such that*

$$\sup_{x \in K} |D_x^\beta D_\xi^\alpha p(x, \xi)| \leq C_{\alpha,\beta,K}(1 + |\xi|)^{m-|\alpha|}.$$

Definition 1.1.5 (Pseudo-differential operator)**.** *A pseudo-differential operator B (PsDO for short) of order m on Ω is a continuously linear map from $C_0^\infty(\Omega)$ to C^∞ of the form*

$$Bu(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} p(x, \xi)\hat{u}(\xi)d\xi, \quad \text{for } u \in C_0^\infty(\Omega), \text{ and } p \in S^m(\Omega), \tag{1.1.2}$$

which can be extended to a continuously linear map from $\mathcal{E}'(\Omega)$ to $D'(\Omega)$. We shall generally denote the map in (1.1.2) *by $p(x, D)$,*

$$p(x, D)u(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} p(x, \xi)\hat{u}(\xi)d\xi, \quad u \in C_0^\infty(\Omega).$$

And we denote the set of pseudo-differential operators of order m on Ω by $\Psi^m(\Omega)$:

$$\Psi^m(\Omega) = \{p(x, D) : \ p \in S^m(\Omega)\}.$$

Example 1.1.3. *For $s \in \mathbb{R}$, the function $(x, \xi) \to (1+|\xi|^2)^{s/2}$ belongs to $S^s(\mathbb{R}^n)$, and hence the operator Λ^s defined by $\Lambda^s f(x) = \mathcal{F}^{-1}((1 + |\xi|^2)^{s/2} f(\xi))$ belongs to $\Psi^s(\mathbb{R}^n)$, where $\mathcal{F}^{-1}$ is the Fourier inverse transformation.*

Lemma 1.1.1 (c.f. [18])**.** *If $P \in \Psi^m(\Omega)$ and $Q \in \Psi^{m'}(\Omega)$, then*
(1) *$PQ \in \Psi^{m+m'}(\Omega)$ and*

$$\sigma_{PQ} = \sigma_P \cdot \sigma_Q \quad \big(mod\ S^{m+m'-1}(\Omega)\big).$$

(2) *$[P, Q] \in \Psi^{m+m'-1}(\Omega)$ and*

$$\sigma_{[P,Q]} = \frac{1}{2\pi i}\{\sigma_P, \sigma_Q\} \quad \big(mod\ S^{m+m'-2}(\Omega)\big),$$

where $[\cdot, \cdot]$ is the Lie bracket in (1.1.1) *and the Poisson bracket $\{\sigma_P, \sigma_Q\}$ defined as follows:*

$$\{\sigma_P, \sigma_Q\} = \sum_{i=1}^{n} \big(\frac{\partial\sigma_P}{\partial\xi_i}\frac{\partial\sigma_Q}{\partial x_i} - \frac{\partial\sigma_Q}{\partial\xi_i}\frac{\partial\sigma_P}{\partial x_i}\big).$$

Definition 1.1.6 (Strongly elliptic)**.** *We say that a symbol $p \in S^m(\Omega)$, or its corresponding operator $p(x, D)$, is strongly elliptic if for every compact $K \subset \Omega$ there are positive constants c, C such that*

$$Re\ p(x, \xi) \geq c(1 + |\xi|^2)^{m/2} \text{ for } x \in K \text{ and } |\xi| \geq C. \tag{1.1.3}$$

Lemma 1.1.2 (Gårding's inequality, c.f. [18])**.** *Supppose $p \in S^m(\Omega)$ satisfies* (1.1.3)*, for any $\varepsilon > 0$, any $s < m/2$, and a open subset V with compact closure in Ω, there are $c > 0$ and $C \geq 0$ depending on V, such that*

$$Re\ \langle p(x, D)u, u\rangle \geq (c - \varepsilon)\|u\|^2_{m/2} - C\|u\|^2_s, \quad u \in C_0^\infty(V).$$

Lemma 1.1.3 (Paley-Wiener theorem, c.f. [48])**.** *$g(\zeta)$ is the Fourier-Laplace transform of a function $f(x) \in C_0^\infty(\mathbb{R}^n)$ with supp $f \subset \{x \in \mathbb{R}^n, |x| \leq A\}$ if and only if for any $N \in \mathbb{N}^+$ there is a constant C_N such that*

$$|g(\zeta)| \leq C_N e^{A|Im\ \zeta|}/(1 + |\zeta|)^N.$$

Lemma 1.1.4 (c.f. [18])**.** *For the Sobolev space $H^s(\mathbb{R}^n)$ and $H^s_{loc}(\Omega)$, we have*
(1) *Every distribution with compact support belongs to $H^s(\mathbb{R}^n)$ for some $s \in \mathbb{R}$.*
(2) *$f \in H^s_{loc}(\Omega)$ if and only if $\varphi f \in H^s(\mathbb{R}^n)$ for every $\varphi \in C_0^\infty(\Omega)$. Moreover $H^s(\mathbb{R}^n) \subset H^s_{loc}(\Omega)$ for every open subset $\Omega \subset \mathbb{R}^n$.*

Proposition 1.1.1. *If X satisfies Hörmander's condition on Ω. Then for any $K \subset\subset \Omega$ and $s \in \mathbb{R}$, there exists $C > 0$ such that*

$$\|u\|_{1+s}^2 \le C\big(\sum_{|\alpha|\le Q} \|X_\alpha u\|_s^2 + \|u\|_s^2\big), \text{ for all } u \in C_0^\infty(K), \tag{1.1.4}$$

where Q is the Hörmander index of X on Ω, $\alpha = (\alpha_1, \cdots, \alpha_k) \in \mathbb{N}^k$, $0 \le \alpha_i \le m$, X_α is the k-th repeated commutators.

Proof: By Hörmander's condition, for any $x_0 \in \Omega$, there exists $r(x_0)$, such that

$$\sum_{|\alpha|\le r(x_0)} |X_\alpha(x_0, \xi)| > 0, \ \xi \neq 0.$$

Since $\sum_{|\alpha|\le r(x_0)} |X_\alpha(x_0, \xi)|$ is a first order positive homogeneous function of ξ, then for a small neighborhood $O(x_0)$ of x_0, it holds that

$$1 + \sum_{|\alpha|\le r(x_0)} |X_\alpha(x, \xi)|^2 \ge C_0(1 + |\xi|^2),$$

where $(x, \xi) \in O(x_0) \times \mathbb{R}^n$, $C_0 > 0$. Since K is compact, thus we can choose a finite number of small open sets $O(x_1), \cdots, O(x_l)$ which can cover K. Also Q is the Hörmander index means that $r(x) \le Q$ for any $x \in \Omega$, then for some constant $C > 0$, we have

$$1 + \sum_{|\alpha|\le Q} |X_\alpha(x, \xi)|^2 \ge C(1 + |\xi|^2),$$

where $(x, \xi) \in K \times \mathbb{R}^n$. So $1 + \sum\limits_{|\alpha|\le Q} X_\alpha^2$ is strongly elliptic. Then Gårding's inequality (Lemma 1.1.2) implies the estimate (1.1.4). □

Proposition 1.1.2. *For any $K \subset\subset \Omega$ and $l \in \mathbb{N}^+$. Then there exist $C > 0$ and $\varepsilon(l) \in \big(0, 1/2^l\big)$ such that*

$$\sum_{|\alpha|\le l} \|X_\alpha u\|_{\varepsilon(l)-1+s}^2 \le C\big(\|Lu\|_s^2 + \|u\|_s^2\big), \text{ for all } u \in C_0^\infty(K), \tag{1.1.5}$$

where $\alpha = (\alpha_1, \cdots, \alpha_k) \in \mathbb{N}^k$, $0 \le \alpha_i \le m$, X_α is the k-th repeated commutators.

Proof: We prove (1.1.5) by induction. For $|\alpha| = 1$, we need to prove

$$\sum_{j=1}^m \|X_j u\|_s^2 \le C\big(\|Lu\|_s^2 + \|u\|_s^2\big). \tag{1.1.6}$$

Since X_j is a real vector fields, then it is self-adjoint and $X_j^* = -X_j + a_j(x)$, here $a_j(x) \in C^\infty$. Thus

$$(Lu, u) = (\sum_{j=1}^m X_j^2 u, u) = -\sum_{j=1}^m \|X_j u\|_0^2 + (\sum_{j=1}^m X_j u, a_j u).$$

Then

$$\sum_{j=1}^m \|X_j u\|_0^2 \le C_1\big(|(Lu, u)| + \|u\|_0^2\big) \le C\big(\|Lu\|_0^2 + \|u\|_0^2\big). \tag{1.1.7}$$

This means that (1.1.6) holds for $s = 0$.

Next, for $s \neq 0$, let Λ^s be a PsDO with the symbol $(1 + |\xi|^2)^{s/2}$, then by Lemma 1.1.1 and direct calculations, we know

$$[X_j, \Lambda^s] \in \Psi^s(\Omega)), j = 1, 2, \cdots, m;$$

$$[L, \Lambda^s] = \sum_{j=1}^m R_j^s X_j + R_0^s,\ R_j^s \in \Psi^s(\Omega), j = 0, 1, \cdots, m.$$

Let $v = \Lambda^s u$. Then from $u \in C_0^\infty(K)$ and Paley-Wiener theorem (Lemma 1.1.3), we have $v \in C_0^\infty(K')$ for any $K \subset K' \subset\subset \Omega$. Next, from (1.1.7), then

$$\begin{aligned}\sum_{j=1}^m \|X_j u\|_s^2 &\le \sum_{j=1}^m \big(\|\Lambda^{-s} X_j v\|_s^2 + \|[X_j, \Lambda^{-s}]v\|_s^2\big)\\ &\le \sum_{j=1}^m \|X_j v\|_0^2 + C\|v\|_0^2 \le C\big(|(Lv, v)| + \|u\|_s^2\big)\\ &\le C\big(|(\Lambda^s L u, \Lambda^s u)| + \sum_{j=1}^m |(R_j^s X_j u, \Lambda^s u)| + |(R_0^s u, \Lambda^s u)| + \|u\|_s^2\big)\\ &\le C\big(\|Lu\|_s^2 + \varepsilon \sum_{j=1}^m \|X_j u\|_s^2 + C_\varepsilon \|u\|_s^2\big).\end{aligned} \tag{1.1.8}$$

Taking ε small such that $C\varepsilon \le 1/2$, then (1.1.8) implies (1.1.6).

Suppose $|\alpha| = k$ and $0 < \varepsilon(k) \le 1/2^k$, we have

$$\sum_{|\alpha| \le k} \|X_\alpha u\|_{\varepsilon(k)-1+s}^2 \le C\big(\|Lu\|_s^2 + \|u\|_s^2\big), \quad \text{for all } u \in C_0^\infty(K). \tag{1.1.9}$$

Then for α satisfying $|\alpha| = k+1$, we seek $\varepsilon(k+1)$ such that (1.1.5) is true. Let $\alpha = \alpha_1 + \alpha'$ with $|\alpha_1| = 1$ and $|\alpha'| = k$, that means

$$X_\alpha = [X_j, X_{\alpha'}],\ j = 1, 2, \cdots, m.$$

Then

$$\begin{aligned}\|X_\alpha u\|_{\varepsilon-1}^2 &= (X_\alpha u, \Lambda^{2\varepsilon-2} X_\alpha u)\\ &= (X_j X_{\alpha'} u, Tu) - (X_{\alpha'} X_j u, Tu)\\ &\le |(X_{\alpha'} u, T X_j u)| + |X_{\alpha'} u, \tilde{T} u| + |(X_j u, T X_{\alpha'} u)| + |(X_j u, \tilde{T} u)|\\ &\le C\big(\|X_j u\|_0^2 + \|X_{\alpha'} u\|_{2\varepsilon-1}^2 + \|u\|_{2\varepsilon-1}^2 + \|u\|_0^2\big)\end{aligned} \tag{1.1.10}$$

where T and $\tilde{T}$ belong to the PsDO class $\Psi^{2\varepsilon-1}$. Taking $\varepsilon = \varepsilon(k+1) \le \varepsilon(k)/2 < 1/2$, then

$$\|X_{\alpha'} u\|_{2\varepsilon-1}^2 \le \|X_{\alpha'} u\|_{\varepsilon(k)-1}^2, \quad \|u\|_{2\varepsilon-1}^2 \le \|u\|_0^2. \tag{1.1.11}$$

So (1.1.7), (1.1.9), (1.1.10) and (1.1.11) imply that (1.1.5) holds for $|\alpha| = k+1$ and $s = 0$. This means

$$\sum_{|\alpha| \le k+1} \|X_\alpha u\|_{\varepsilon(k+1)-1}^2 \le C\big(\|Lu\|_0^2 + \|u\|_0^2\big), \quad \text{for all } u \in C_0^\infty(K).$$

Next, for $s \neq 0$, similar to the above estimates, using the commutator technique, we can also obtain

$$\sum_{|\alpha|\leq k+1} \|X_\alpha u\|^2_{\varepsilon(k+1)-1+s} \leq C\big(\|Lu\|_s^2 + \|u\|_s^2\big), \text{ for all } u \in C_0^\infty(K).$$

These complete the proof of Proposition 1.1.2. □

Proof of Theorem 1.1.1: (1.1.4) and (1.1.5) imply that for any $K \subset\subset \Omega$, there exist $C > 0$ and $\varepsilon(Q) \in \big(0, 1/2^Q\big)$ such that

$$\|u\|^2_{\varepsilon(Q)+s} \leq C\big(\|Lu\|_s^2 + \|u\|_s^2\big), \quad \text{for all } u \in C_0^\infty(K). \tag{1.1.12}$$

For any $\varphi \in C_0^\infty(\Omega)$, if $u \in \mathcal{D}'(\Omega)$, then $\varphi u \in \mathcal{E}'(\Omega)$. Lemma 1.1.4 tells us that there exists $s_0 \in \mathbb{R}$ such that

$$\varphi u \in H^{s_0}(\mathbb{R}^n) \subset H^{s_0}_{loc}(\Omega). \tag{1.1.13}$$

On the other hand, $Lu \in C^\infty(\Omega)$ means that for any $s \in \mathbb{R}$, it holds that

$$Lu \in H^s_{loc}(\Omega). \tag{1.1.14}$$

Then combining (1.1.12), (1.1.13) and (1.1.14), we have $\varphi u \in H^{s_0+\varepsilon(Q)}_{loc}(\Omega)$. Repeating the process above, we know $\varphi u \in H^s_{loc}(\Omega)$, for any $s \in \mathbb{R}$, which implies $\varphi u \in C^\infty(\Omega)$. Next, by the arbitrariness of $\varphi \in C_0^\infty(\Omega)$, we can deduce that $u \in C^\infty(\Omega)$. □

1.1.2 Sharp Sub-elliptic Estimate

From the discussions above, if X satisfies the Hörmander's condition with the Hörmander index Q, then the sub-elliptic estimate (1.1.12) holds with the index $\varepsilon(Q) \leq 1/2^Q$. However, the number $1/2^Q$ is not optimal. In fact, we have the following sharp sub-elliptic estimate (cf. [16] and [25]).

Theorem 1.1.2. *If the system of real smooth vector fields X satisfies the Hörmander's condition on Ω. Then*

$$\big\||\nabla|^{1/Q} u\big\|^2_{L^2(\Omega)} \leq C(Q)(\|Xu\|^2_{L^2(\Omega)} + \widetilde{C}(Q)\|u\|^2_{L^2(\Omega)}), \quad \textit{for all } u \in C_0^\infty(\Omega). \tag{1.1.15}$$

Here Q is the Hörmander index of X on Ω, $|\nabla|^{1/Q}$ is a PsDO with the symbol $|\xi|^{1/Q}$, $C(Q) > 0$ and $\widetilde{C}(Q) \geq 0$ depending on Q.

Remark 1.1.1. *After we introduce the sub-elliptic metrics (in Section 1.2) and the weighted Sobolev spaces (in Section 1.3), we shall give a brief proof of Theorem 1.1.2 in Section 1.3 below. Also we can point out that the number* $1/Q$ *in* (1.1.15) *is optimal, one can refer to [51] for the details.*

1.2 Geometry Induced by Vector Fields

1.2.1 Sub-elliptic Metric

Let $\Omega \subset \mathbb{R}^N$ be a connected open domain, and let $X = \{X_1, X_2, \cdots, X_m\}$ be C^∞ real vector fields defined in the neighborhood of Ω (or defined on $\mathbb{R}^N$).

Definition 1.2.1 (Sub-unit curve)**.** *For any $\delta > 0$, let the sub-unit curve $C_1(\delta)$ be the class of absolutely continuous mappings $\varphi : [0, 1] \to \Omega$ which satisfy*

$$\varphi'(t) = \sum_{j=1}^{m} c_j(t) X_j(\varphi(t)), \text{ a.e. with } |c_j(t)| \le \delta. \tag{1.2.1}$$

Definition 1.2.2 (Carnot-Carathéodory metric)**.** *We define the Carnot-Carathéodory distance $d_1(x, y)$ as follows:*

$$d_1(x, y) = \begin{cases} \inf\{\delta > 0 : \ \exists \varphi \in C_1(\delta) \text{ with } \varphi(0) = x, \varphi(1) = y.\}, \\ +\infty, \text{ if there doesn't exist } \varphi \in C_1(\delta) \text{ with } \varphi(0) = x, \varphi(1) = y. \end{cases}$$

Moreover, we say that d_1 is the Carnot-Carathéodory metric if $d_1 < \infty$.

Remark 1.2.1. *If we only have the single vector field $X = \{\partial_{x_1}\}$ in $\mathbb{R}^2$, then $d_1(x - y) = |x - y|$ if x and y lie on a line parallel to the x_1 axis; otherwise $d_1(x, y) = \infty$. On the other hand, if $X = \{\partial_{x_1}, \cdots, \partial_{x_N}\}$ in $\mathbb{R}^N$, then d_1 is the Euclidean metric.*

Theorem 1.2.1 (Rashevski-Chow's connectivity theorem, c.f.[12, 49])**.** *Let the system of vector fields X satisfy the Hörmander's condition on an open connected set $\Omega \subset \mathbb{R}^N$. Then for every couple of points $x, y \in \Omega$ there exists an absolutely continuous curve φ contained in Ω and jointing x to y, such that φ is composed by integral curves of the X_i's.*

Remark 1.2.2. *Rashevski-Chow's connectivity theorem tells us that if the system of vector fields X satisfies the Hörmander's condition, then d_1 is the Carnot-Carathéodory metric. However, the Carnot-Carathéodory distance above might be well defined even if the vector fields do not satisfy the Hörmander's condition (e.g. some cases for the vector fields to be infinitely degenerate).*

Suppose X satisfies the Hörmander's condition, we introduce the sub-elliptic metric and the metric balls induced by X.

Let

$$X^{(1)} = \{X_1, \cdots, X_m\}, \quad X^{(2)} = \{[X_1, X_2], \cdots, [X_{m-1}, X_m]\}, \quad \text{etc.}$$

so that the components of $X^{(k)}$ are the commutators of length k. Let $Y_1, \cdots, Y_q$ be some enumeration of the components of $X^{(1)}, \cdots, X^{(k)}$. If Y_i is an element of $X^{(j)}$, we say Y_i has formal degree $d(Y_i) = j$.

Definition 1.2.3 (Sub-elliptic curve)**.** *For any $\delta > 0$, let the sub-elliptic curve $C_2(\delta)$ be the class of absolutely continuous mappings $\varphi : [0, 1] \to \Omega$ which satisfy*

$$\varphi'(t) = \sum_{j=1}^{q} c_j(t) Y_j(\varphi(t)), \text{ a.e. with } |c_j(t)| \le \delta^{d_j}, \tag{1.2.2}$$

where $Y_1, \cdots, Y_q$ are some enumeration of the components of $X^{(1)}, \cdots, X^{(k)}$ for some $k \in \mathbb{N}^+$ satisfying $span\{Y_i\}_{i=1}^q = \mathbb{R}^N$ and $d_j \ge 1$ is the formal degree of Y_j.

Remark 1.2.3. *$span\{Y_i\}_{i=1}^q = \mathbb{R}^N$ means that for any two points in Ω which can be connected by a sub-elliptic curve.*

Definition 1.2.4 (Sub-elliptic distance)**.** *We define the sub-elliptic distance $\rho(x, y)$ as follows:*

$$\rho(x, y) = \inf\{\delta > 0 : \ \exists \ \varphi \in C_2(\delta) \text{ with } \varphi(0) = x, \varphi(1) = y.\}.$$

Remark 1.2.4. *$\rho(x,y)$ is called the sub-elliptic metric on Ω.*

Proposition 1.2.1. *If $K \subset\subset \Omega$ is any compact set, then there are constants C_1, C_2 so that if $x, y \in K$,*

$$C_1|x-y| \le \rho(x,y) \le C_2|x-y|^{1/Q}, \tag{1.2.3}$$

where Q is the Hörmander index of X on Ω.

Proof: Let $K \subset\subset \Omega$ be an arcwise connected compact set. There is a constant C so that if $x, y \in K$, there is an absolutely continuous function $\varphi : [0,1] \to \Omega$ with $\varphi(0) = x, \varphi(1) = y$ and $|\varphi'(t)| \le C|x-y|$ for all t. Since $Y_1, \cdots, Y_q$ span $\mathbb{R}^N$, then we can write

$$\varphi'(t) = \sum_{j=1}^{q} c_j(t) Y_j(\varphi(t)),$$

with $|c_j(t)| \le C'|\varphi'(t)| \le C''|x-y| = C''(|x-y|^{1/d_j})^{d_j}$. Observe that $d_j \le Q$, it follows that

$$\rho(x,y) \le C|x-y|^{1/Q}.$$

Conversely, if $x, y \in K$ and $\rho(x,y) = \delta$, then there exists $\varphi \in C_2(2\delta)$ with $\varphi(0) = x, \varphi(1) = y$ and $\varphi'(t) = \sum_{j=1}^{q} a_j(t) Y_j(\varphi(t))$ with $|a_j(t)| \le (2\delta)^{d_j}$. Since the components of every Y_j are uniformly bounded in Ω, it follows that

$$|\varphi'(t)| \le C \sum_{j=1}^{q} (2\delta)^{d(Y_j)} \le C'\delta.$$

Hence

$$|x-y| = \Big|\int_0^1 \varphi'(t)dt\Big| \le C'\delta = C'\rho(x,y).$$

□

Theorem 1.2.2. *If X satisfies the Hörmander's condition on Ω, then the metrics d_1 and ρ are locally equivalent.*

Lemma 1.2.1 (c.f. Lemma 2.20 in [44])**.** *Let $w \in \Omega$, and w has a neighborhood U so that if x_1 and x_∞ are in U with $\rho(x_1, x_\infty) < \varepsilon$, then the following two conclusions hold:*

(a) *There exists $x_2 \in U$ with $d_1(x_1, x_2) < C\varepsilon$, and $\rho(x_2, \infty) < C\varepsilon^{1+1/Q}$.*

(b) *Given $y \in U$ there is a number $\eta(y) > 0$ so that if $|z-y| < \eta(y)$, we have $d_1(y,z) < C|z-y|^{1/Q}$.*

Proof of Theorem 1.2.2: It is obvious that $\rho \le d_1$. On the other hand, near a point $w \in \Omega$ we can choose U a neighborhood of w which is so small such that we may use the result of Lemma 1.2.1 on U. Let $x = x_1$, and y be in U with $\rho(x,y) = \delta$. We apply Lemma 1.2.1 with $x_1 = x$, $x_\infty = y$ and obtain a point x_2 with

$$d_1(x_1, x_2) < C\delta \text{ and } \rho(x_2, y) < C\delta^{1+1/Q} < \delta/2,$$

if $C\delta^{1/Q} < 1/2$. We can then apply Lemma 1.2.1 again with $\varepsilon = \delta/2$ to obtain x_3 so that $\rho(x_2, x_3) < C\delta/2$, $\rho(x_3, y) < \delta/4$. In general, given $x = x_1, x_2, \cdots, x_j$ we can find x_{j+1} so that $\rho(x_j, x_{j+1}) < C\delta/2^{j-1}$ and $\rho(x_{j+1}, y) < \delta/2^j$. Moreover d_1 satisfies the triangle inequality so $d_1(x, x_j) < C\delta$. By part (b) of Lemma 1.2.1 we see that if j is sufficiently large, $d_1(x_j, y) < \delta$. Using the triangle inequality again for d_1 completes the proof. □

Remark 1.2.5. *To prove Lemma 1.2.1, we need Campbell-Hausdorff formula and the generalization of the Campbell-Hausdorff formula. For more details about Campbell-Hausdorff formula, one can refer to [24, 44] and here we omit these.*

1.2.2 Sub-elliptic Balls and Doubling Property

It follows from Proposition 1.2.1 that the sub-elliptic metric $\rho : \Omega \times \Omega \to [0, \infty)$ is continuous. Then we can define the following sub-elliptic ball.

Definition 1.2.5 (Sub-elliptic balls). *We can define a sub-elliptic ball $B(x, \delta)$ on Ω by*

$$B(x, \delta) = \{y \in \Omega : \rho(x, y) < \delta\}.$$

Now, we give a characterization of finitely degenerate vector fields in the view of geometry.

Proposition 1.2.2 (c.f.[16]). *The following statements are equivalent:*
(1) *X satisfies the Hörmander's condition with the Hörmander index Q.*
(2) *There exists $C > 0$ such that*

$$B_E(x, \rho) \subset B_X(x, C\rho^Q), \text{ for any } x \in \Omega, \rho > 0.$$

Here $B_E(x, \rho)$ is an ordinary Euclidean ball of radius ρ about x, $B_X(x, C\rho^Q)$ is a sub-elliptic ball of radius $C\rho^Q$ induced by X.

For each N-tuple of integers $I = (i_1, \cdots, i_N)$ with $1 \le i_j \le q$, set

$$\lambda_I(x) = det(Y_{i_1}, \cdots, Y_{i_N})(x).$$

(If $Y_{i_j} = \sum_{k=1}^N a_{jk}(x)(\partial/\partial x_k)$), then $det(Y_{i_1}, \cdots, Y_{i_N})(x) = det(a_{jk}(x))$. We also set $d(I) = d(Y_{i_1}) + \cdots + d(Y_{i_N})$ and then we define

$$\Lambda(x, \delta) = \sum_I |\lambda_I(x)| \delta^{d(I)},$$

where the sum is over all N-tuples. Now we state the known result on the volumes of the balls $B(x, \delta)$.

Theorem 1.2.3 (Nagel-Stein-Wainger's theorem of metric balls). *For every compact set $K \subset\subset \Omega$, there are constants C_1 and C_2 so that for all $x \in K$,*

$$0 < C_1 \le \frac{|B(x, \delta)|}{\Lambda(x, \delta)} \le C_2 < +\infty.$$

Example 1.2.1. *Let us consider the Grushin vector fields:*

$$X_1 = \partial_x; \; X_2 = x\partial_y, \; \text{in } \mathbb{R}^2.$$

To make $\lambda_I(x) \neq 0$, we can only have two choices as follows:

$$Y_{i_1} = \partial_x, Y_{i_2} = x\partial_y; \; \text{or} \; Y_{i_1} = \partial_x, Y_{i_2} = \partial_y.$$

For the case $Y_{i_1} = \partial_x, Y_{i_2} = x\partial_y$, then $\lambda_I(x) = x$ and $d(I) = 2$. For the case $Y_{i_1} = \partial_x, Y_{i_2} = \partial_y$, then $\lambda_I(x) = 1$ and $d(I) = 3$. So the above theorem states that

$$C_1(\delta^3 + \delta^2|x|) \le |B\big((x, y), \delta\big)| \le C_2(\delta^3 + \delta^2|x|).$$

In particular, the balls of center $(0, y_0)$ have volume comparable to δ^3, while the balls of center (x_0, y_0) with large x_0 and small radius δ have volume comparable to δ^2.

Definition 1.2.6 (Doubling property). *We say that (Ω, ρ) satisfies doubling property if for any $K \subset\subset \Omega$, there exist $r_0 > 0$ and $C \geq 1$ such that*

$$|\tilde{B}(x, 2r)| \leq C|\tilde{B}(x, r)|,$$

where

$$x \in K, r \leq r_0, \tilde{B}(x, r) = \{y \in \Omega, \rho(x, y) < r\}.$$

Remark 1.2.6. *If X satisfies the Hörmander's condition, since Λ in Theorem 1.2.3 is a polynomial in δ of fixed degree, it follows immediately from Theorem 1.2.3 that (Ω, ρ) is doubling. On the other hand, if $\rho < 1$, then from Proposition 1.2.2(2),we can directly deduce that $|B_X(x, 2\rho)| \leq C|B_X(x, \rho)|$. That means that (Ω, ρ) is doubling.*

In order to describe the sub-elliptic ball $B(x, \delta)$ more precisely, we need following concepts.

Definition 1.2.7 (Métivier index). *If X satisfies the Hörmander's condition on Ω with the Hörmander index Q, then for each $1 \leq j \leq Q$ and $x \in \Omega$, we denote $V_j(x)$ as the subspace of the tangent space $T_x(\Omega)$ which is spanned by the vector fields $\{X_J\}$ with $|J| \leq j$. the Métivier index at $x \in \Omega$ is defined as*

$$\nu(x) = \sum_{j=1}^{Q} j(\nu_j(x) - \nu_{j-1}(x)), \text{ here } \nu_0 = 0. \tag{1.2.4}$$

where $\nu_j(x)$ is the dimension of $V_j(x)$.

Moreover, if the dimension of $V_j(x)$ is constant ν_j for a neighborhood of each $x \in \Omega$. Then we say that X satisfies the Métivier's condition on Ω and $\nu = \nu(x)$ is the Métivier index on Ω.

Remark 1.2.7. *If X satisfies the Hörmander's condition on Ω, then the volume of the ball with small radius r induced by the sub-elliptic metric satisfies*

$$|B(x, r)| \approx r^{\nu}, \text{ for all } x \in \Omega, \tag{1.2.5}$$

where ν is the Métivier index.

Now let us introduce the following definition.

Definition 1.2.8 (Hausdorff dimension). *Let Ω be an open connected bound domain in $\mathbb{R}^n$ with the metric ρ. The Hausdorff dimension of Ω is defined as*

$$\inf\{\alpha > 0; H^{\alpha}(\Omega) = 0\} = \sup\{\alpha > 0; H^{\alpha}(\Omega) = +\infty\}, \tag{1.2.6}$$

where

$$H^{\alpha}(\Omega) := \lim_{\delta \to 0} H^{\alpha}_{\delta}(\Omega), \; H^{\alpha}_{\delta}(\Omega) := \inf \Big\{ \sum_{i=1}^{\infty} diam(\Omega_i)^{\alpha}; \Omega \subset \bigcup_{i=1}^{\infty} \Omega_i, diam(\Omega_i) < \delta \Big\},$$

and

$$diam(\Omega_i) = \max\{\rho(x, y); \; x, y \in \Omega_i\}.$$

Example 1.2.2. (1) *Let Ω be an open connected bound domain in $\mathbb{R}^n$ with the Euclidean metric ρ. Then Hausdorff dimension of Ω is n.*

(2) *Let Ω be an open connected bound domain in $\mathbb{R}^{2N+1}$ with the sub-elliptic metric ρ induced by Heisernberg Group. Then in this case the Métivier's condition is satisfied, and Hausdorff dimension of Ω is $2N+2$, which is the same with the Métivier index ν.*

(3) *Ω is an open connected bound domain in $\mathbb{R}^n$ with the sub-elliptic metric ρ induced by $X = (\partial_{x_1}, \partial_{x_2}, \cdots, \partial_{x_{n-1}}, x^l\partial_{x_n})$. Then Hausdorff dimension of Ω is $n+l$.*

Remark 1.2.8. *In [38], the author proved that if X satisfies the Hörmander's condition and Métivier condition on Ω, the Métivier index ν on Ω equals to the Hausdorff dimension of Ω. Moreover, if the Métivier condition does not hold on Ω for X, then the Hausdorff dimension might be the general Métivier index $\bar{\nu}$ (see Definition 1.3.4 below).*

Now we give a brief proof of Nagel-Stein-Wainger's theorem (For the detail proof please see [44]).

First, we introduce a simplification of notation. If $x \in E \subset\subset \Omega$ and $I = (i_1, \cdots, i_N)$ are fixed, we shall relabel the vector fields $\{Y_j\}$ be setting $U_j = Y_{i_j}$, $1 \le j \le N$, and let V_j, $1 \le j \le q-N$, being some enumeration of the remaining vector fields. If $u = (u_1, \cdots, u_N) \in \mathbb{R}^N$ and $v = (v_1, \cdots, v_{q-N}) \in \mathbb{R}^{q-N}$, we let

$$u \cdot U + v \cdot V = \sum_{j=1}^{N} u_j U_j + \sum_{j=1}^{q-N} v_j V_j,$$

and

$$\Phi_v(u) = \exp(u \cdot U + v \cdot V)(x).$$

For $v \in \mathbb{R}^{q-N}$, we let $z = \exp(v \cdot V)(x)$ and introduce one more family of balls

$$B_I(x, z, \delta) = \{y \in \Omega; y = \exp(u \cdot U + v \cdot V)(x), \text{ with } |u_j| < \delta^{d(U_j)}\}. \tag{1.2.7}$$

Thus $B_I(x, z, \delta)$ is exactly the image, under the map Φ_v of the box $\{u \in \mathbb{R}^N; |u_j| < \delta^{d(U_j)}\} = Q(\delta)$.

Proposition 1.2.3. *Let $E \subset\subset \Omega$ be compact. There exist constants $0 < \eta_2 < \eta_1 < 1$ so that if $x \in E$, $|v_j| < \eta_2 \delta^{d(V_j)}$, $1 \le j \le q-N$ and $\delta > 0$ there exists an N-tuple $I = (i_1, \cdots, i_N)$ with the following properties:*

(1) *Φ_v is global one-to-one for $|u_j| < (\eta_1 \delta)^{d(U_j)}$.*

(2) *Let $J\Phi_v$ denote the Jacobian of Φ_v, then on the box $Q(\eta_1\delta)$, we have*

$$\frac{1}{4}|\lambda_I(x)| \le |J\Phi_v| \le 4|\lambda_I(x)|. \tag{1.2.8}$$

(3) *Let $z = \exp(v \cdot V)(x)$, then*

$$B(x, \eta_2\delta) \subset B_I(x, z, \delta) \subset B(x, \delta). \tag{1.2.9}$$

Proof of Theorem 1.2.3: First, Proposition 1.2.3(1) and (2) show that $B_I(x, z, \eta_1\delta)$ is the image under the one to one mapping Φ_v of the box $Q(\eta_1\delta)$ and the Jacobian of this mapping is bounded between two constant multiples of $\lambda_I(x)$, then it follows that

$$|B_I(x, z, \eta_1\delta)| \approx |\lambda_I(x)||Q(\eta_1\delta)| \approx |\lambda_I(x)|\delta^{d(I)}. \tag{1.2.10}$$

Moreover, Proposition 1.2.3(3) tells us that $B(x, \eta_2\delta) \subset |B_I(x, z, \eta_1\delta)| \subset B(x, \delta)$, it follows that

$$|B(x, \delta)| \approx \sum_I |\lambda_I(x)|\delta^{d(I)}. \tag{1.2.11}$$

Then (1.2.10) and (1.2.11) imply the result of Theorem 1.2.3. □

Next, we prove Proposition 1.2.3.

Lemma 1.2.2. *Let $E \subset\subset \Omega$ be compact. There exist constants $\eta_1 \in (0,1)$ so that if $x \in E$ and $\delta > 0$ there exists an N-tuple $I = (i_1, \cdots, i_N)$ satisfying*

$$|\lambda_I(x)|\delta^{d(I)} \geq \eta_2 \max_J |\lambda_J(x)|\delta^{d(J)}. \tag{1.2.12}$$

Proof: Let $E \subset\subset \Omega$ be compact let $x \in E$. Let $I_0 = I_0(x_0)$ be an N-tuple such that $d(I_0)$ is minimal among all N-tuple J with $\lambda_J(x_0) \neq 0$, and such that

$$|\lambda_{I_0}(x_0)| = \max_{d(J)=d(I_0)} |\lambda_J(x_0)|. \tag{1.2.13}$$

Then there exists δ_0 depending on x_0 such that

$$|\lambda_{I_0(x_0)}|\delta^{d(I_0)} \geq |\lambda_J|\delta^{d(J)}, \tag{1.2.14}$$

for all δ, $0 < \delta \leq \delta_0$, and all N-tuple J.

Since the Jacobian of the exponential map is the identity at the origin, we can find an open set $W = W_{x_0}$ in Ω containing x_0 so that the mapping

$$(u_1, \cdots, u_n) \mapsto \Phi_v(u_1, \cdots, u_n) = \exp(u \cdot U + v \cdot V)(x)$$

is globally one to one on $|u| < \delta_0$ for all x in W, $|v| < \delta_0$. Also for some $W' \subset\subset W$, we know

$$|\lambda_{I_0}(x)|\delta^{d(I_0)} \geq \frac{1}{2}|\lambda_J(x)|\delta^{d(J)}, \tag{1.2.15}$$

for all $0 < \delta \leq \delta_0$, all N-tuple J and $x \in W'$. Next, Choosing a finite open covering $W_{x_1}, \cdots, W_{x_l}$ of E, taking $\bar{\delta} = \inf_{j=1,\cdots,l} \delta_i$ and

$$\bar{I} = \{I_i; |\lambda_{I_i}(x)|\delta^{d(I_i)} = \inf_{j=1,\cdots,l} |\lambda_{I_j}(x)|\delta^{d(I_j)},$$

then

$$|\lambda_{\bar{I}}(x)|\delta^{d(\bar{I})} \geq \frac{1}{2}|\lambda_J(x)|\delta^{d(J)}, \tag{1.2.16}$$

for all $0 < \delta \leq \bar{\delta}$, all N-tuple J and $x \in E$. □

Proof of Proposition 1.2.3(2): First, we have

$$J\Phi_v = det\big(d\Phi_v(\frac{\partial}{\partial u_1}), \cdots, d\Phi_v(\frac{\partial}{\partial u_n})\big). \tag{1.2.17}$$

However

$$|det(U_1, \cdots, U_N)\big(\Phi_v(u)\big)| = |\lambda_I\big(\Phi_v(u)\big)|. \tag{1.2.18}$$

By the technique of exponential mapping (see [24, 44]), we can prove that

$$\frac{1}{2}|\lambda_I(x)| \leq |\lambda_I\big(\Phi_v(u)\big)| \leq 2|\lambda_I(x)|, \tag{1.2.19}$$

and

$$Z_j = \sum_{l=1}^{n}(\delta_{jl} + b_{jl})U_l, \tag{1.2.20}$$

where $Z_j = d\Phi_v(\frac{\partial}{\partial u_j})$, $|b_{jl}| < \chi\delta^{d(U_l)-d(U_j)}$ and χ can be taken sufficiently small. Then from (1.2.20), we can solve the U_l in the term of the Z_j. Then (1.2.17), (1.2.18), (1.2.19), (1.2.20) imply (1.2.8). □

Lemma 1.2.3. *For $|v_j| < \delta^{d(V_j)}$, if $z = \exp(v \cdot V)(x)$, then*

$$B(z, \eta\delta) \subset B_I(x, z, \delta), \tag{1.2.21}$$

where $x \in E$ and the n-tuple I satisfy (1.2.12).

Proof: Let $y \in B(z, \eta\delta)$. Then there is an absolutely continuous map $\varphi : [0, 1] \to \Omega$ with $\varphi(0) = z$, $\varphi(1) = y$ and

$$\varphi'(t) = \sum_{j=1}^{q} b_j(t) Y_j(\varphi(t)),$$

with $|b_j(t)| \le (\eta\delta)^{d_j}$. We can also assume that the map φ is one to one.

Let $\mathcal{F}$ be the set of numbers $s_0 \in [0, 1]$ such that there exists an absolutely continuous mapping $\theta : [0, s_0] \to \mathbb{R}^n$ such that $|\theta_j(s)| \le (\delta/2)^{d(V_j)}$ and

$$\varphi(s) = \exp\,(\sum_{j=1}^{N} \theta_j(s) U_j + v \cdot V)(x),\ 0 \le s \le s_0.$$

Since the mapping $(u_1, \cdots, u_n) \mapsto \exp(u \cdot U + v \cdot V)(x)$ is locally one to one on $\{u \in \mathbb{R}^n; |u_j| < \delta^{d(U_j)}\}$, then we let $\bar{s} = \sup\{s_0 \in \mathcal{F}\}$, and it can be deduced that $\bar{s} \le 1$.

The mapping $\Phi_v(u_1, \cdots, u_n) = \exp(u \cdot U + v \cdot V)(x)$ is locally one to one, and since the map φ and θ are one to one on $[0, \bar{s}]$, and $\varphi(s) = \Phi_v(\theta(s))$. It follows that Φ_v is actually globally one-to-one on some small neighborhood of the image $\theta[0, \bar{s}]$. Thus we can think of the components of the inverse map $(\psi_1, \cdots, \psi_n)$ as being well defined functions in some neighborhood of $\theta([0, \bar{s}])$.

Suppose $\bar{s} < 1$, then for some j_0 we must have

$$\psi_{j_0}(\bar{s}) = (\delta/2)^{d(U_{j_0})}. \tag{1.2.22}$$

On the other hand, for any j_0 we have

$$\begin{aligned} |\psi_{j_0}(\bar{s})| = |\psi_{j_0}(\bar{s}) - \psi_{j_0}(0)| &= \Big| \int_0^{\bar{s}} \frac{d}{ds} \psi_{j_0}(s) ds \Big| \\ &= \Big| \int_0^{\bar{s}} \sum_{j=1}^{q} b_j(s) Y_j(\varphi(s)) \psi_{j_0}(s) ds \Big| \\ &\le (\eta\delta)^{d_j} C \delta^{d(U_{j_0}) - d(U_j)} = C\eta^{d_j} \delta^{d(U_{j_0})}. \end{aligned} \tag{1.2.23}$$

Then if η is small enough such that $C\eta^{d_j} < (1/2)^{d(U_{j_0})}$, then (1.2.23) is contradictive with (1.2.22). This means $\bar{s} = 1$. And then

$$y = \varphi(1) = \exp\,(\sum_{j=1}^{N} \theta_j(1) U_j + v \cdot V)(x),$$

with $|\theta_j(1)| \le \delta^{d(U_j)}$ and so $y \in B_I(x, z, \delta)$. □

Proof of Proposition 1.2.3(3): From the definitions of $B(x, \delta), B_I(x, z, \delta)$ and Lemma 1.2.2, it is obvious that if $|v|_j < \delta^{d(V_j)}$ for $1 \le j \le q - n$ then we have the inclusions

$$B_I(x, z, \delta) \subset B(x, \delta), \tag{1.2.24}$$

where $z = \exp(v \cdot V)(x)$. Next, for the above $\eta > 0$ in Lemma 1.2.3, taking $\eta_1, \eta_2 \in (0,1)$ such that $\eta_1 \le (\eta/2)^{d(U_j)}$ and $\eta_2 \le (\eta/2)^{d(V_j)}$ $(\le \eta/2)$. This means that if $|u_j| < \eta_1 \delta^{d(U_j)}$ and $|v_j| < \eta_2 \delta^{d(V_j)}$, then by the definition of ρ, we have $\rho(x,z) < \eta\delta/2$. Then

$$B(x, \eta\delta/2) \subset B(z, \eta\delta). \tag{1.2.25}$$

Then (1.2.21), (1.2.24) and (1.2.25) imply (1.2.9). □

Lemma 1.2.4. *Suppose for some δ,* (1.2.12) *holds for I_0, I_1. For the above η_1 and η_2. If $|\eta_j| < (\eta_2\delta)^{d(V_j)}$, then it holds that*

$$B_{I_1}(x, z, \eta_2\delta) \subset B_{I_0}(x, 0, \eta_1\delta) \subset B_{I_1}(x, z, \delta). \tag{1.2.26}$$

Proof: It is obvious from the above proofs for the relations of metric balls. □

Proof of Proposition 1.2.3(1): For $x \in K$, taking I_0 satisfies (1.2.16), from the definition of exponential mapping, we know that the mapping

$$(u_1, \cdots, u_n) \mapsto \Phi_v(u_1, \cdots, u_n) = \exp(u \cdot U + v \cdot V)(x) \tag{1.2.27}$$

is globally one to one if $x \in K$, $|u| < \delta_0$ and $|v| < \delta_0$, where K is a compact subset of W containing x. In particular, it follows that the image of any simply connected set is simply connected.

Choosing a sequence of n-tuples $I_1, \cdots, I_l$ and $\delta_0 > \delta_1 > \cdots > \delta_l > 0$ so that for $\delta_{j+1} \le \delta \le \delta_j$, $0 \le j \le l-1$,

$$|\lambda_{I_j}(x)|\delta^{d(I_j)} \ge \frac{1}{2}|\lambda_J(x)|\delta^{d(J)},$$

and for $0 < \delta \le \delta_l$,

$$|\lambda_{I_l}(x)|\delta^{d(I_l)} \ge \frac{1}{2}|\lambda_J(x)|\delta^{d(J)}.$$

We may clearly assume $d(I_{j+1}) < d(I_j)$. In particular, no n-tuple occurs twice, and l is at most the total number of allowable n-tuples. The choice of the particular n-tuple of course may depend on x.

Let $\Phi_v^{(1)}$ be the mapping (1.2.27) associated to the n-tuple I_1. If $\Phi_v^{(1)}$ were not globally one-to-one on $|u_j| < (\eta_2\delta)^{d(U_j)}$, there would be a line segment L in the box

$$\{u \in \mathbb{R}^n; |u_j| < (\eta_2\delta)^{d(U_j)}\},$$

which $\Phi_v^{(1)}$ maps to a closed curve in $B_{I_1}(x, z, \eta_2\delta)$, where $z = \exp(v \cdot V)(x)$. However, this curve can be deformed to a point in $B_{I_0}(x, 0, \eta_1\delta)$ and hence by Lemma 1.2.4 it can be deformed to a point in $B_{I_1}(x, z, \delta)$, which is impossible. Thus $\Phi_v^{(1)}$ is globally one-to-one.

By repeating this argument l times for successive series of N-tuples I_{j+1} and I_j, we can prove that the mapping $\Phi_v(u)$ is globally one-to-one for $|u_j| < (\eta_1\delta)^{d(U_j)}$. □

1.3 Weighted Sobolev Spaces and Embedding

1.3.1 The Spaces $H^{k,p}_X(\Omega)$ and $S^{k,\alpha}(\Omega)$

Let a system of vector fields $X = \{X_1, X_2, \cdots, X_m\}$ defined on a open bounded domain Ω with smooth boundary $\partial\Omega$. Then, for $k \in \mathbb{N}$, $1 \le p \le +\infty$, we define

$$H^{k,p}_X(\Omega) = \{f \in L^p(\Omega) \mid X^J f \in L^p(\Omega),\ \forall |J| \le k\}, \tag{1.3.1}$$

where $J = (j_1, \cdots, j_l)$ with $1 \le j_i \le m$, $X^J = X_{j_1} X_{j_2} X_{j_3} \cdots X_{j_{l-1}} X_{j_l}$, $|J| = l$. Also we define the norm in $H_X^{k,p}(\Omega)$ to be

$$\|f\|_{H_X^{k,p}(\Omega)} = \Big(\sum_{|J| \le k} \|X^J f\|^p_{L^p(\Omega)} \Big)^{1/p}.$$

We also denote by $H_X^k(\Omega) = H_X^{k,2}(\Omega)$.

Theorem 1.3.1. *For $k \in \mathbb{N}$ and $1 \le p < +\infty$, then the space $H_X^{k,p}(\Omega)$ is a Banach space.*

Proof: Let $J = (j_1, \cdots, j_s)$ with $1 \le j_i \le m$ for $i = 1, \cdots, s$ and denote by $X^{J,*}$ the adjoint operator of X^J. Then

$$\begin{aligned} H_X^{k,p}(\Omega) = &\big\{ f \in L^p(\Omega) \mid \exists\, g_J \in L^p(\Omega) \text{ such that} \\ &\int_\Omega f \cdot X^{J,*} \varphi dx = \int_\Omega g_J \varphi dx, \quad \text{for any } \varphi \in C_0^\infty(\Omega),\ |J| \le k \big\}. \end{aligned} \tag{1.3.2}$$

Suppose $\{u_j\}$ to be a Cauchy sequence of $H_X^{k,p}(\Omega)$, then $\{X^J u_j\}$, for $|J| \le k$, are all Cauchy sequence in $L^p(\Omega)$. Hence there exists $u_J \in L^p(\Omega)$ such that $X^J u_j \to u_J$ in $L^p(\Omega)$. On the other hand

$$\int_\Omega u_j X^{J,*} \varphi dx = \int_\Omega X^J u_j \varphi dx,\ \forall \varphi \in C_0^\infty(\Omega),\ |J| \le k.$$

Let $j \to \infty$, we have that there is $u_0 \in L^p$, such that

$$\int_\Omega u_0 X^{J,*} \varphi dx = \lim_{j \to \infty} \int_\Omega X^J u_j \varphi dx,\ \forall \varphi \in C_0^\infty(\Omega),\ |J| \le k,$$

which proves $u_0 \in H_X^{k,p}(\Omega)$, $X^J u_0 = u_J$ and $\|u_j - u_0\|_{H_X^{k,p}(\Omega)} \to 0$. $\square$

Now we denote by $H_{X,0}^{k,p}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $H_X^{k,p}(\Omega)$.

Definition 1.3.1 (Characteristic and non-characteristic)**.** *If $L = \sum_{|\alpha| \le k} a_\alpha(x) D_x^\alpha$ is a linear differential operator of order k on $\Omega \subset \mathbb{R}^n$, here $D_x^\alpha = D_{x_1}^{\alpha_1} \cdots D_{x_n}^{\alpha_n}$ and $D_{x_j} = \frac{1}{\sqrt{-1}} \partial_{x_j}$. The characteristic form of L at $x \in \Omega$ is the homogeneous polynomial of degree k on $\mathbb{R}^n$ defined by*

$$\chi_L(x, \xi) = \sum_{|\alpha| = k} a_\alpha(x) \xi^\alpha, \qquad (\xi \in \mathbb{R}^n).$$

A nonzero vector ξ is called characteristic for L at x if $\chi_L(x, \xi) = 0$, and the set of all such ξ is called the characteristic variety of L at x and is denoted by $Char_x(L)$:

$$Char_x(L) = \{\xi \ne 0 : \chi_L(x, \xi) = 0\}.$$

A hypesurface S is called characteristic for L at $x \in S$ if the normal vector $\nu(x)$ to S at x is in $Char_x(L)$, and S is called non-characteristic if it is not characteristic at any point.

Theorem 1.3.2. *For $k \in \mathbb{N}$ and $1 \le p < +\infty$, if $\partial\Omega$ is C^∞ and non characteristic for the system X, then $H_{X,0}^{k,p}(\Omega)$ is well-defined, and a Banach space.*

Proof: For simplification, we only prove the case $k = 1$, and for $k \neq 1$, the proof is similar.

For the well-definedness, we need to prove the existence of trace for $v \in H^{1,p}_X(\Omega)$. We know that the trace problem is a local problem, so after the localization and straightened, we transfer the problem to the case : $v \in L^p(\mathbb{R}^n_+)$, $\partial_{x_n} v \in L^p(\mathbb{R}^n_+)$ with support of v is a subset of $\{|(x', x_n)| < c, x_n \geq 0\}$, of course we can take the smooth function approximate to v, then we have

$$v(x', x_n) - v(x', c) = \int_c^{x_n} \partial_t v(x', t) dt,$$

which proves that

$$\|v(\cdot, x_n)\|_{L^p(\mathbb{R}^{n-1})} \leq c \|\partial_{x_n} v\|_{L^p(\mathbb{R}^{n-1})}, \tag{1.3.3}$$

for all $0 \leq x_n \leq c$. This shows that the trace $v(x', 0) \in L^p(\mathbb{R}^{n-1})$.

We shall prove now $H^{1,p}_{X,0}(\Omega)$ is a closed subspace of $H^{1,p}_X(\Omega)$. Let $\{v_j\}$ be a Cauchy sequence of $H^{1,p}_{X,0}(\Omega)$. Since it is also a Cauchy sequence of $H^{1,p}_X(\Omega)$, there exists a limit $v_0 \in H^{1,p}_X(\Omega)$, and so it suffices to show that $v|_{\partial\Omega} = 0$. Applying (1.3.3) to $v_j - v_0$, we have

$$\|v_j(\cdot, 0) - v_0(\cdot, 0)\|_{L^p(\mathbb{R}^{n-1})} \leq c \|\partial_{x_n}(v_j - v_0)\|_{L^p(\mathbb{R}^{n-1})},$$

which implies $\|v_0(\cdot, 0)\|_{L^p(\mathbb{R}^{n-1})} = 0$. We have proved that $H^{1,p}_{X,0}(\Omega)$ is a Banach space. $\square$

Example 1.3.1. *If $X = (\partial x_1, \cdots, \partial x_{n-1}, x_1 \partial x_n)$ defined on a ball B_n in $\mathbb{R}^n$ with $\{x_1 = 0\} \cap \partial B_n \neq \varnothing$. Then we can verify that ∂B_n is non-characteristic for X.*

If $X = \{X_1, X_2, \cdots, X_m\}$ satisfies Hörmander's condition on a bounded open domain Ω with Hörmander index $Q > 1$, then for $0 < \alpha < 1$, we define

$$S^\alpha(\Omega) = \{f \in C(\Omega) \cap L^\infty(\Omega); [f]_{\alpha,\Omega} = \sup_{x,y \in \Omega} \frac{|f(x) - f(y)|}{\rho(x,y)^\alpha} < +\infty\}, \tag{1.3.4}$$

where $\rho(x, y)$ is the sub-elliptic distance.

For $k \in \mathbb{N}, 0 \leq \alpha < 1$, we define

$$S^{k,\alpha}(\Omega) = \{f \in S^\alpha(\Omega); X^J f \in S^\alpha(\Omega), \forall |J| \leq k\}, \tag{1.3.5}$$

where $S^0(\Omega) = C(\Omega) \cap L^\infty(\Omega)$. Set

$$[u]_{k,\Omega} = \sup_{x \in \Omega, |J| = k} |X^J u(x)|, \ [u]_{k,\alpha,\Omega} = \sup_{|J| = k} [X^J u(x)]_{\alpha,\Omega}.$$

We define the norm in $S^{k,\alpha}(\Omega)$ by

$$\|u\|_{S^{k,\alpha}(\Omega)} = \sum_{j=0}^k [u]_{j,\Omega} + [u]_{k,\alpha,\Omega}.$$

Replacing Ω by $\bar{\Omega}$ in (1.3.4) and (1.3.5), we can also define $S^\alpha(\bar{\Omega})$ and $S^{k,\alpha}(\bar{\Omega})$.

Lemma 1.3.1. *Let X satisfy Hörmander's condition on Ω. Then*

$$S^\alpha(\Omega) \subset C^{\alpha/Q}(\Omega) \text{ and } S^{kQ,0}(\Omega) \subset C^k_{Lip}(\Omega).$$

for $0 \leq \alpha < 1$ and $k \in \mathbb{N}$, where C^λ is the usual Hölder space, $C^k_{Lip}(\Omega)$ is the Lipschitz space and Q is the Hörmander index of X on Ω.

Proof: It is obvious by the definitions of $S^\alpha(\Omega)$, $S^{kQ,0}(\Omega)$ and the result of Proposition 1.2.1. $\square$

Remark 1.3.1. *From Lemma 1.3.1, we know that*

$$S^{k,\alpha}(\Omega) \subset C^{(k+\alpha)/Q}(\Omega), \text{ for } k \in \mathbb{N} \text{ and } 0 \le \alpha < 1.$$

Similarly, we can also have

$$S^{k,\alpha}(\bar{\Omega}) \subset C^{(k+\alpha)/Q}(\bar{\Omega}), \text{ for } k \in \mathbb{N} \text{ and } 0 \le \alpha < 1.$$

Theorem 1.3.3. *For $k \in \mathbb{N}$ and $0 \le \alpha < 1$, the space $S^{k,\alpha}(\Omega)(S^{k,\alpha}(\bar{\Omega}))$ is a Banach space.*

Proof: For $k = 0$, we assume that $\{f_j\} \subset S^\alpha(\Omega)$ is a Cauchy sequence. Thus $\|f_j\|_{S^\alpha(\Omega)} \le M < +\infty$. Using Lemma 1.3.1, $\{f_j\} \subset S^\alpha(\Omega)$ is equicontinuous, so there exists $f_0 \in C(\Omega)$ such that $f_j \to f_0$ in $C(\Omega)$. For $0 < \alpha < 1, x \ne y, x, y \in \Omega$, we have

$$\begin{aligned}\frac{|f_0(x) - f_0(y)|}{\rho(x,y)^\alpha} &\le \frac{|f_0(x) - f_j(x)|}{\rho(x,y)^\alpha} + \frac{|f_j(x) - f_j(y)|}{\rho(x,y)^\alpha} + \frac{|f_j(y) - f_0(y)|}{\rho(x,y)^\alpha} \\ &\le 2 + [f_j]_{\alpha,\Omega} \le 2 + M.\end{aligned} \tag{1.3.6}$$

That proves $f_0 \in S^\alpha(\Omega)$. For $k = 1$, similarly we have $f_m \to f_0 \in C(\Omega)$ and $X_j f_m \to \tilde{f}_j$ in $C(\Omega)$. Here f_0 and $\tilde{f}_j \in S^\alpha(\Omega)$. Thus we need to prove $\tilde{f}_j(x) = X_j f_0(x)$ for all $x \in \Omega$. If $X_j(x) = 0$, then $X_j f_m(x) \to \tilde{f}_j(x) = X_j f_0(x) = 0$. Assume now $X_j(x) \ne 0$, and denotes by $\phi(t)$ the integral curve of X_j with $\phi(0) = x$, then for small $|t|$,

$$f_m(\phi(t)) - f_m(\phi(0)) = \int_0^t X_j f_m(\phi(s)) ds.$$

Because f_m and $X_j f_m$ are all uniformly convergent, we have

$$f_0(\phi(t)) - f_0(\phi(0)) = \int_0^t \tilde{f}_j(\phi(s)) ds.$$

So $(d/dt) f_0(\phi(t))|_{t=0} = \tilde{f}_j(x)$, but $(d/dt) f_0(\phi(t))|_{t=0} = X_j f_0(x)$, which proves $X_j f_0 = \tilde{f}_j$. The general cases can be proved in the same way. $\square$

Proposition 1.3.1 (Interpolation inequality)**.** *Suppose $j+\beta < k+\alpha$, $j, k \in \mathbb{N}$, $0 \le \alpha, \beta \le 1$ and $u \in S^{k,\alpha}(\Omega)$. Then for any $\varepsilon > 0$, there exists a constant $C_\varepsilon = C(\varepsilon, j, k, \Omega)$ such that*

$$\|u\|_{S^{j,\beta}(\Omega)} \le \varepsilon \|u\|_{S^{k,\alpha}(\Omega)} + C_\varepsilon \|u\|_{L^\infty(\Omega)}.$$

Proof: It is sufficient to prove the following interpolation inequality for seminorms:

$$[u]_{j,\beta,\Omega} \le \varepsilon [u]_{k,\alpha,\Omega} + C_\varepsilon \|u\|_{L^\infty(\Omega)}. \tag{1.3.7}$$

We prove (1.3.7) by induction, and suppress the index Ω, take $d > 0$ small enough, such that

$$\Omega_d = \{x \in \Omega; \rho(x, \partial\Omega) > d\} \ne \emptyset.$$

(a) Let $j = 1$, $k = 2$, $\alpha = \beta = 0$, we need to prove

$$[u]_1 \le \varepsilon [u]_2 + C_\varepsilon \|u\|_{L^\infty(\Omega)}. \tag{1.3.8}$$

By definition $[u]_1 = \sup_j \sup_{x\in\Omega} |X_j u(x)|$. For $u \in S^2(\Omega)$ fixed, there exists j_0, and $x_0 \in \bar{\Omega}$ such that $[u]_1 = |X_{j_0}u(x_0)|$. Let $\mu \in (0, 1/2)$ to be chosen; we first consider the case $B(x_0, \mu d) \subset \Omega$. For $[u]_1 \neq 0$, we have $X_{j_0}(x_0) \neq 0$. Let $\varphi(t)$ be the integral curve of X_{j_0} with $\varphi(0) = x_0$, take $\mu d \geq \delta \geq \mu d/2$, such that $\varphi(\delta) = x_2 \in B(x_0, \mu d)$. Then

$$u(x_0) - u(x_2) = u(\varphi(0)) - u(\varphi(\delta)) = X_{j_0}u(\varphi(\theta))\delta.$$

Let $\varphi(\theta) = \bar{x} \in B(x_0, d)$. Then

$$|X_{j_0}u(\bar{x})| \leq |u(x_0) - u(x_2)|/\delta \leq \frac{4}{\mu d}|u|_0.$$

On the other hand, there exists $\varphi_1 \in C_2(\mu d)$ such that $\varphi_1(0) = x_0$ and $\varphi_1(1) = \bar{x}$, hence

$$\begin{aligned} X_{j_0}u(x_0) - X_{j_0}u(\bar{x}) &= X_{j_0}u(\varphi_1(0)) - X_{j_0}u(\varphi_1(1)) \\ &= \int_0^1 \sum_{j=1}^m a_j(t) X_j(X_{j_0}u(\varphi_1(t)))dt. \end{aligned}$$

So

$$|X_{j_0}u(x_0)| \leq \frac{4}{\mu d}|u|_0 + \mu d \sum_{j=1}^m \sup_{y\in\Omega} |X_j X_{j_0} u(y)|.$$

Take $\mu > 0$ small enough such that $\mu d m \leq \varepsilon$, we have proved (1.3.8) in the case $B(x_0, \mu d) \subset \Omega$.

For the case $\rho(x_0, \partial\Omega) < \mu d$, we consider $B(x_1, \mu d) \subset \Omega$, where $x_1 \in \Omega_{\mu d} \cap B(x_0, \mu d)$. If $X_{j_0}(x_1) = 0$, we have

$$X_{j_0}u(x_0) - X_{j_0}u(x_1) = \int_0^1 \sum_{j=1}^m a_j(t) X_j(X_{j_0}u(\varphi_1(t)))dt,$$

hence,

$$|X_{j_0}u(x_0)| \leq \mu d \sum_{j=1}^m \sup_{x\in\Omega} |X_j X_{j_0} u(y)| \leq \mu d m [u]_2.$$

If $X_{j_0}(x_1) \neq 0$, as above, there exists $\bar{x} \in B(x_1, \mu d)$ such that $|X_{j_0 u(\bar{x})}| \leq \frac{4}{\mu d}|u|_0$ and $\rho(\bar{x}, x_0) \leq 2\mu d$, then we can obtain (1.3.8) as above.

Let $j = k = 2, \beta = 0, \alpha > 0$, and $u \in S^{2,\alpha}(\Omega)$. By definition we have $[u]_2 = \sup_{ij} \sup_{x\in\Omega} |X_i X_j u(x)| = |X_{i_0} X_{j_0} u(x_0)|$. As in point (a), we consider only the case $x_0 \in \Omega_{\mu d}$. Assume that $X_{i_0}(x_0) \neq 0$ and $X_{j_0}u(x_0) - X_{j_0}u(x_2) = X_{i_0}X_{j_0}u(\bar{x})\delta$ with $x_0, \bar{x} \in B(x_1, \mu d)$ and $\mu d \geq \delta \geq (\mu d)/2$. Then

$$|X_{i_0} X_{j_0} u(\bar{x})| \leq \frac{4}{\mu d}[u]_1,$$

and so

$$\begin{aligned} |X_{i_0}X_{j_0}u(x_0)| &\leq |X_{i_0}X_{j_0}u(\bar{x})| + |X_{i_0}X_{j_0}u(x_0) - X_{i_0}X_{j_0}u(\bar{x})| \\ &\leq \frac{4}{\mu d}[u]_1 + (\mu d)^\alpha [u]_{2,\alpha}. \end{aligned}$$

Using (a) we have proved $[u]_2 \leq \varepsilon[u]_{2,\alpha} + C_\varepsilon |u|_0$ with $\varepsilon = 2(\mu d)^\alpha$.

The other cases are similar. $\square$

Here, we give the proof of Theorem 1.1.2.

Definition 1.3.2 (Operator of type λ)**.** *Let $\lambda > 0$, T is called an operator of type λ, if it is defined by a distribution kernel $T(x,y)$ which satisfies the following estimate*

$$|X^\alpha T(x,y)| \leq C_\alpha \rho(x,y)^{\lambda-|\alpha|}|B(x,\rho(x,y))|^{-1}. \tag{1.3.9}$$

Proposition 1.3.2. *Suppose T is an operator of type 1. Then T maps $L^2_0(\Omega)$ to $W^{1/Q,2}(\Omega)$, here $L^2_0(\Omega) = \{u \in L^2(\Omega); u|_{\partial\Omega} = 0,\ a.e.\}$.*

Proposition 1.3.3. *For all $f \in C_0^\infty(\Omega)$, there exist operators $T_0, T_1, \cdots, T_m$, of type 1, such that*

$$f(x) = \sum_{i=1}^m T_i X_i f(x) + T_0 f(x). \tag{1.3.10}$$

Proof of Theorem 1.1.2: For $u \in H^1_{X,0}(\Omega)$, we know $u \in L^2_0(\Omega)$ and $X_j u \in L^2_0(\Omega)$. Then from Proposition 1.3.2 and Proposition 1.3.3, we have $u \in W^{1/Q,2}(\Omega)$. $\square$

The proof of Proposition 1.3.2 is similar to the proof of Proposition 1.3.6 below, and we shall give a brief proof later (one can also refer to [19] and [51] for the detail proof).

The proof of Proposition 1.3.3 depends on the following result:

Proposition 1.3.4 (Fundamental solutions)**.** *If the real smooth system of vector fields $X = \{X_1, X_2, \cdots, X_m\}$ satisfies Hörmander's condition on a open bounded domain Ω, then there exists a distribution function $G(x,y)$ for $(x,y) \in \Omega \times \Omega$ satisfying*

$$LG(x,y) := \sum_{i=1}^m X_i^2 G(x,y) = \delta_x(y), \tag{1.3.11}$$

i.e. for any $f \in L^2(\Omega)$, we define $u(x) = \int_\Omega G(x,y)f(y)dy$, then it holds that $Lu(x) = f(x)$. Moreover, $G(x,y)$ satisfies, for all $(x,y) \in \Omega \times \Omega$,

$$G(x,y) = G(y,x), \text{ and } |X_{j_1} \cdots X_{j_s} G(x,y)| \leq C_s \rho(x,y)^{2-s}|B(x,\rho(x,y))|^{-1}. \tag{1.3.12}$$

Remark 1.3.2. *The proof of Proposition 1.3.4 is omitted here, and one can refer [52] for the details. Also,* (1.3.9) *and* (1.3.12) *show that $G(x,y)$ is the type 2 and $X_j G(x,y)$ is the type 1 for $j = 1, \cdots, m$.*

Proof of Proposition 1.3.3: From (1.3.11),

$$\begin{aligned} f(x) &= \sum_{i=1}^m \int_\Omega X_i^2(x) G(x,y) f(y) dy \\ &= \sum_{i=1}^m \int_\Omega T_i(x,y) X_i(x) f(y) dy + \int_\Omega T_0(x,y) f(y) dy \\ &= \sum_{i=1}^m T_i X_i f(x) + T_0 f(x), \end{aligned} \tag{1.3.13}$$

where $T_i(x,y) = X_i(x)G(x,y)$ and $T_0(x,y) = \sum_{i=1}^m [X_i(x), T_i(x,y)]$.

Next, by the definition of type of $\lambda > 0$ and the properties of the fundamental solutions $G(x,y)$, it is obvious that operators $T_0, T_1, \cdots, T_m$ are type 1. $\square$

1.3.2 Weighted Sobolev Embedding

Theorem 1.3.4 (Weighted Sobolev embedding theorem I). *Let Ω be a bounded open domain of $\mathbb{R}^n$. Assume that X satisfies the Hörmander's condition on Ω. Then, we have the continuous embedding $H^{k,p}_{X,0}(\Omega) \subset W^{k/Q,p}(\Omega)$ for all $k \geq 1$, $p \geq 1$ and there exists $C = C(p, \Omega, Q)$ such that*

$$\|u\|_{W^{k/Q,p}(\Omega)} \leq C\|u\|_{H^{k,p}_X(\Omega)},$$

for all $u \in H^{k,p}_{X,0}(\Omega)$, where Q is the Hörmander index of X on Ω and $W^{s,p}(\Omega)$ is the usual Sobolev space.

Lemma 1.3.2. *Suppose T is an operator of type λ. Then T maps $L^p_0(\Omega)$ to $W^{\lambda/Q,p}(\Omega)$, $1 < p < +\infty$, here $L^p_0(\Omega) = \{u \in L^p(\Omega); u|_{\partial\Omega} = 0,\ a.e.\}$.*

Remark 1.3.3. *Proposition 1.3.2 is the special case of Lemma 1.3.2 for $\lambda = 1$ and $p = 2$. The detail proof of Lemma 1.3.2 can be also found in Theorem 12 of [51].*

Proposition 1.3.5 (Representation theorem of $H^{k,p}_X(\Omega)$). *For all $f \in H^{k,p}_X(\Omega)$, there are T_α, which are the operators of type k, such that*

$$f(x) = \sum_{|\alpha| \leq k} T_\alpha X^\alpha f(x). \tag{1.3.14}$$

Proof of Theorem 1.3.4: For all $u \in H^{k,p}_{X,0}(\Omega)$, then $X^\alpha u \in L^p_0(\Omega)$ for all $|\alpha| \leq k$, then from Lemma 1.3.2 and Proposition 1.3.5, we have $u \in W^{k/Q,p}(\Omega)$. □

Proof of Proposition 1.3.5: Suppose the function $a(x) \in C^\infty_0(\mathbb{R}^n)$. Similar to the proof of Proposition 1.3.3, there exists operators $T_0, T_1, \cdots, T_m$, of type 1, such that for any $f \in C^\infty_0(\mathbb{R}^n)$,

$$a(x)f(x) = \sum_{j=1}^{m} T_j X_j f(x) + T_0 f(x). \tag{1.3.15}$$

Then taking an open covering $\{\Omega_i\}_{i=1}^l$ of $\bar{\Omega}$, and $a_i \in C^\infty_0(\Omega_i)$ with $\sum_{i=1}^{l} a_i(x) = 1$ for $x \in \bar{\Omega}$. Then we have

$$a_i(x)f(x) = \sum_{j=1}^{m} T^i_j X_j f(x) + T^i_0 f(x) \text{ in } \Omega_i. \tag{1.3.16}$$

Hence for any $f \in C^\infty(\bar{\Omega})$, $f(x) = \sum_{j=1}^m T_j X_j f(x) + T_0 f(x)$, where $T_j = \sum_{i=1}^l T^i_j$, $j = 0, 1, \cdots, m$ are the operators of type 1. Since $C^\infty(\Omega)$ is dense in $H^{1,p}_X(\Omega)$, then

$$f(x) = \sum_{j=1}^{m} T_j X_j f(x) + T_0 f(x), \text{ for all } f \in H^{1,p}_X(\Omega). \tag{1.3.17}$$

Thus we have proved the proposition for $k = 1$. Suppose that it is true for $k-1$, we need to prove the result in case of k. Taking $f \in H^{k,p}_X(\Omega)$, we have $X^\alpha f \in H^{1,p}_X(\Omega)$ for all $|\alpha| \leq k-1$. Therefore

$$f = \sum_{|\alpha| \leq k-1} T_\alpha X^\alpha f = \sum_{|\alpha| \leq k-1} T_\alpha \big(\sum_{j=1}^{m} T_j X_j + T_0 \big) X^\alpha f, \tag{1.3.18}$$

where $T_\alpha T_j$, $j = 0, 1, \cdots, m$, are the operators of type k. The proof of Proposition 1.3.5 is completed. □

Corollary 1.3.1. *Let Ω be a bounded open C^∞ domain of $\mathbb{R}^n$. Assume that X satisfies the Hörmander's condition in Ω. Then we have continuous embedding*

$$H^{k,p}_{X,0}(\Omega) \subset \begin{cases} L^{nQp/(Qn-kp)}(\Omega), & \text{for } kp < nQ, \\ C^m(\bar{\Omega}), & \text{for } k/Q - n/p > m \geq 0, \end{cases}$$

where Q is the Hörmander index of X on Ω.

Proof: This is direct result by Theorem 1.3.4 and the classical Sobolev embedding in $W^{k,p}(\Omega)$. □

Remark 1.3.4. *Comparing Corollary 1.3.1 with the classical embedding*

$$W^{k,p}_0(\Omega) \subset L^{np/(n-kp)}(\Omega), \text{ for } kp < n,$$

we only replace n in the classical Sobolev embedding by nQ in Corollary 1.3.1. In fact, this index is not optimal. Next, we shall give the optimal embedding results. Follow Definition 1.2.7, we have

Theorem 1.3.5 (Weighted Sobolev embedding theorem II)**.** *Suppose that X satisfies the Hörmander's condition and the Métivier condition. Let $1 < p \leq +\infty$, then*

(1) *if $kp < \nu$, then $H^{k,p}_X(\Omega)$ is continuously embedded in $L^{\nu p/(\nu-kp)}(\Omega)$, i.e.*

$$H^{k,p}_X(\Omega) \subset L^{\nu p/(\nu-kp)}(\Omega), \tag{1.3.19}$$

where ν is the Métivier index of X on Ω.

(2) *If $kp > \nu$, then $H^{k,p}_{X,0}(\Omega)$ is continuously embedded in $S^{l,(k-\nu/p)-l}(\bar{\Omega})$, where*

$$l = [k - \nu/p] = \max\{j \in \mathbb{N}^+; j \leq [k - \nu/p]\}.$$

Remark 1.3.5. *Observe that $Q + n - 1 \leq \nu \leq Qn$, here n is the topology dimension of Ω, Q is the Hörmander index and ν is the Métivier index of X on Ω. Thus $kp < \nu$ implies $Qnp/(Qn - kp) \leq \nu p/(\nu - kp)$. That means the result of Theorem 1.3.5 is sharp than the result of Corollary 1.3.1.*

Remark 1.3.6. *Let $1 < p \leq +\infty$. If $kp < \nu$ and $1 < q < \nu p/(\nu - kp)$, then similar to the classical Sobolev compactly embedding (cf. [14]), we can prove that the embedding $H^{k,p}_X(\Omega) \hookrightarrow L^q(\Omega)$ is compact.*

Proposition 1.3.6. *Assuming that T is an operator of type $\lambda > 0$, if $0 < \lambda q < \nu$, then $T : L^q(\Omega) \to L^p(\Omega)$ is continuous, where $1/p = 1/q - \lambda/\nu > 0$ and $1 < p, q < +\infty$.*

Proof of Theorem 1.3.5(1): For all $u \in H^{k,p}_X(\Omega)$, then $X^\alpha u \in L^p(\Omega)$ for all $|\alpha| \leq k$, Proposition 1.3.5 and Proposition 1.3.6 imply $u \in L^{\nu p/(\nu-kp)}(\Omega)$. □

Proposition 1.3.7. *Suppose that T is an operator of type $\lambda > 0$, if $\nu < \lambda q$, then $T : L^q_0(\Omega) \to S^{\lambda-\nu/q}(\bar{\Omega})$ is continuous.*

Proof of Theorem 1.3.5(2): For $u \in H^{k,p}_{X,0}(\Omega)$, then $X^\alpha u, u \in L^p_0(\Omega)$, $|\alpha| \leq k$. Using (1.3.9) and Proposition 1.3.7, for all $\nu < kp$, we have $u \in S^{k-\nu/p}(\bar{\Omega}) = S^{l,(k-\nu/p)-l}(\bar{\Omega})$, where $l = [k - \nu/p] = \max\{j \in \mathbb{N}^+; j \leq [k - \nu/p]\}$. This proves Theorem 1.3.5(2). □

In order to prove Proposition 1.3.6, we need to introduce the following results.

Definition 1.3.3 (Weak $L^p(\Omega,\mu)$ space)**.** *Let (Ω,Σ,μ) be a measure space, and f be a measurable function with real or complex values on Ω. The distribution function of f is defined for $t>0$ by*

$$\lambda_f(t)=\mu\{x\in\Omega:|f(x)|>t\}.$$

Then the function f is said to be in the space $L^{p,w}(\Omega,\mu)$ (the weak $L^p(\Omega,\mu)$ space) with $1\le p<\infty$, if there is a constant $C>0$ such that, for all $t>0$,

$$\lambda_f(t)\le\frac{C^p}{t^p}.$$

The best constant C for this inequality is the $L^{p,w}$-norm of f, and is denoted by

$$\|f\|_{p,w}=\sup_{t>0}\ t\lambda_f^{\frac{1}{p}}(t).$$

Remark 1.3.7. *$L^p(\Omega,\mu)\subsetneqq L^{p,w}(\Omega,\mu)$ with $1\le p<\infty$. It is obvious by definition that $L^p(\Omega,\mu)\subset L^{p,w}(\Omega,\mu)$. Next, by direct calculations, the function $\frac{1}{|x|}\in L^{1,w}(\mathbb{R})$, but $\frac{1}{|x|}\notin L^1(\mathbb{R})$.*

The proof of Proposition 1.3.6 depends on the following lemma.

Lemma 1.3.3. *Let k be a measurable function on $\Omega\times\Omega$ such that, for some $r>1$, $k(x,y)$ is weak L^r uniformly in x and y respectively. Then the operator $Af(x)=\int_\Omega k(x,y)f(y)dy$ is bounded from L^q to L^p whenever $\frac{1}{p}=\frac{1}{q}+\frac{1}{r}-1$ and $1<q<p<\infty$.*

Proof: This is Proposition 15.3 in [19], we omit the proof here. □

Proof of Proposition 1.3.6: For $\rho(x,y)$ is small, (1.3.9) implies $|T(x,y)|\le C\rho(x,y)^{\lambda-\nu}$. On the other hand, since $\bar\Omega$ is compact, then from the doubling property we can deduce that $|T(x,y)|\le C\rho(x,y)^{\lambda-\nu}$ for all $x,y\in\Omega$.

Next, we can calculate that $T(x,y)$ is weak $L^{\nu/(\nu-\lambda)}(\Omega)$ uniformly in x and y respectively. Specifically,

$$\begin{aligned}\lambda_T(t)&=\mu\{x\in\Omega:|T(x,y)|>t\}\\&\le\mu\left\{x\in\Omega:\rho(x,y)<(1/t)^{1/(\nu-\lambda)}\right\}\\&\le C(1/t)^{\nu/(\nu-\lambda)}.\end{aligned}\tag{1.3.20}$$

Then $\sup_{t>0}\ t\lambda_T^{(\nu-\lambda)/\nu}(t)$ is bounded.

By using the result in Lemma 1.3.3 with $r=\nu/(\nu-\lambda)>1$, we can then complete the proof of Proposition 1.3.6. □

Now, let us give a proof for Proposition 1.3.2.

Proof of Proposition 1.3.2: Denote $\Lambda=Op\{<\xi>\}$, then from Hörmander's condition we have $\Lambda=\sum_{|\alpha|\le Q}a_\alpha X_\alpha$ (here $Q\ge 2$ is the Hörmander index of X). Thus

$$\|Tu\|_{W^{1/Q,2}(\Omega)}\le C\sum_{|\alpha|\le Q}\|X_\alpha^{1/Q}Tu\|_{L^2(\Omega)}.$$

From the definition, we know the operator $X_\alpha^{1/Q}T$ is the operator of type $1-1/Q$. Thus we choose $p=\frac{2Q\nu}{Q(\nu-2)+2}>2$, then $\frac{1}{p}=\frac{1}{2}-\frac{Q-1}{Q\nu}$ (here ν is the Métivier index). By Hölder inequality, one has

$$\|X_\alpha^{1/Q}Tu\|^2_{L^2(\Omega)}\le\|X_\alpha^{1/Q}Tu\|_{L^p(\Omega)}\cdot\|X_\alpha^{1/Q}Tu\|_{L^q(\Omega)},$$

where $q = \frac{p}{p-1} \in (1, 2)$. Since Ω is bounded, then we have

$$\|X_\alpha^{1/Q} Tu\|_{L^q(\Omega)} \le C_1 \|X_\alpha^{1/Q} Tu\|_{L^2(\Omega)}.$$

Thus, from the result of Proposition 1.3.6 we can deduce that

$$\begin{aligned}\|Tu\|_{W^{1/Q,2}(\Omega)} \le & C \sum_{|\alpha|\le Q} \|X_\alpha^{1/Q} Tu\|_{L^2(\Omega)} \\ \le & C_2 \sum_{|\alpha|\le Q} \|X_\alpha^{1/Q} Tu\|_{L^p(\Omega)}. \\ \le & C_3 \|u\|_{L^2(\Omega)}\end{aligned} \tag{1.3.21}$$

The proof of Proposition 1.3.2 is completed. □

Proof of Proposition 1.3.7: Since the problem is local, we can first suppose that $g \in L^q(\Omega)$ and supp $g \subset B(x_0, R) \subset\subset \Omega$. Then $Tg(x) = \int_\Omega T(x, y)g(y)dy$. Now for $x, x' \in \Omega$, $\rho(x, x') = \delta < 1$, there exists $\xi \in \Omega$ such that $\rho(\xi, x), \rho(\xi, x') < \delta$. Then we have

$$Tg(x) - Tg(x') = \int_{B(\xi,3\delta)} (T(x, y) - T(x, y'))g(y)dy + \int_{\Omega\setminus B(\xi,3\delta)} (T(x, y) - T(x', y))g(y)dy.$$

Since sub-elliptic distance ρ and the C-C distance d_1 are equivalent. Then there is $\alpha(t) \in C_2(2\delta)$, such that

$$\alpha(0) = x, \alpha(1) = x' \text{ and } \rho(x, \alpha(t)), \rho(x', \alpha(t)) < 2\delta,$$

for all $0 \le t \le 1$. Thus

$$T(x, y) - T(x', y) = \sum_{j=1}^m \int_0^1 a_j(t) X_j(\alpha(t))T(\alpha(t), y)dt,$$

with $|a_j(t)| \le 2\delta, j = 1, 2, \cdots, m$. We have also, for $y \in \Omega\setminus B(\xi, 3\delta)$, $\rho(\alpha(t), y) \ge \delta = \rho(x, x')$, and $B(\xi, 3\delta) \subset B(x, 4\delta) \cap B(x', 4\delta)$. Hence

$$\begin{aligned}|Tg(x) - Tg(x')| \le & C \int_{B(x,4\delta)} \rho(x, y)^\lambda |B(x, \rho(x, y))|^{-1} |g(y)|dy \\ & + C \int_{B(x',4\delta)} \rho(x', y)^\lambda |B(x', \rho(x', y))|^{-1} |g(y)|dy \\ & + C \int_0^1 dt \int_{\Omega\setminus B(x,3\delta)} \rho(x', x) \int_0^1 \rho(\alpha(t), y)^{\lambda-1} |B(\alpha(t), \rho(\alpha(t), y))|^{-1} |g(y)|dy \\ =: & I_1 + I_2 + I_3.\end{aligned}$$

The estimates of I_1 and I_2 are similar:

$$\begin{aligned}I_1 \le & C\|g\|_{L^q(\Omega)} \big(\int_{B(x,4\delta)} \rho(x, y)^{(\lambda-\nu)q/(q-1)} dy \big)^{(q-1)/q} \\ \le & C\|g\|_{L^q(\Omega)} \big(\int_{B(x,4\delta)} \rho(x, y)^{(q\lambda-\nu)/(q-1)} |B(x, \rho(x, y))|^{-1} \big)^{(q-1)/q} dy \\ \le & C\delta^{\lambda-\nu/q} \|g\|_{L^q(\Omega)}.\end{aligned}$$

For I_3, we have

$$\begin{aligned} I_3 &\le C\rho(x,x')\int_0^1 dt\int_{\Omega\setminus B(x,3\delta)}\rho(\alpha(t),y)^{\lambda-1-\nu}|g(y)|dy\\ &\le C\rho(x,x')\Big(\int_{\delta<\rho(x,y)<2R}\rho(x,y)^{(\lambda-1-\nu)q/(q-1)}dy\Big)^{(q-1)/q}\|g\|_{L^q(\Omega)}\\ &\le C\rho(x,x')\rho^{\lambda-1-q+\nu(q-1)/q}\|g\|_{L^q(\Omega)}\\ &= C\delta^{\lambda-\nu/q}\|g\|_{L^q(\Omega)}. \end{aligned}$$

We have proved $Tg \in S^{\lambda-\nu/q}(\Omega)$. For $x_0 \in \partial\Omega$ and supp $g \subset B(x_0,R)\cap\Omega$, similar to the estimates above, we have analogous results. □

From Definition 1.2.7, for general cases, we have

Definition 1.3.4. *We define*

$$\bar\nu = \max_{x\in\Omega}\nu(x), \tag{1.3.22}$$

as the general Métivier index on Ω.

Remark 1.3.8. *It is obvious that $\bar\nu = \nu$ if X satisfies the Métivier's condition on Ω.*

Remark 1.3.9. *For more general vector fields X, the result of Theorem 1.3.5 would be also hold if we use the general Métivier index $\bar\nu$ to replace the Métivier index ν. In this case the corresponding Sobolev critical exponent in $H^{k,p}_X(\Omega)$ would be $\bar\nu p/(\bar\nu - kp)$.*

1.4 Boundary-Value Problems

1.4.1 Bony's Maximum Principle

Let

$$L = \sum_{j=1}^m X_j^2(x) + X_0(x) + c(x),$$

where $\{X_j\}_{j=1}^m$ are real smooth vector fields and $c(x) \le 0$ is a C^∞ function defined on Ω.

Theorem 1.4.1 (Bony's maximum principle I, cf. [5])**.** *Suppose that u is a C^2 function defined on Ω, satisfying*

$$Lu \ge 0.$$

Let $Z \in \mathfrak{X}(X_1,\dots,X_m)$ be a vector field and Γ a integral curve for Z. If u attains its non-negative maximum at a point in Γ, then u is constant within Γ.

Corollary 1.4.1 (Bony's maximum principle II)**.** *Let the system of vector fields X satisfy the Hörmander condition on Ω. If $u \in C^2(\Omega)$ and $\triangle_X u \ge 0$, then u can not take its maximum on interior points of Ω expect that it is constant on the connected component of those points.*

Proof: This is the direct result from Rashevski-Chow's connectivity theorem (i.e. Theorem 1.2.1) and Bony's maximum principle I (i.e. Theorem 1.4.1). □

Remark 1.4.1. *The operator L in Corollary 1.4.1 can be "very degenerate" at each point. For example, (denote the coordinate as $(x_0, x_1, \cdots, x_n)$ in $\mathbb{R}^{n+1}$):*

$$L = \frac{\partial^2}{\partial x_0^2} + \big(x_0 \frac{\partial}{\partial x_1} + x_0^2 \frac{\partial}{\partial x_2} + \ldots + x_0^n \frac{\partial}{\partial x_n}\big)^2.$$

In order to prove Theorem 1.4.1, we need the following propositions and lemmas.

Proposition 1.4.1. *Assume Ω is an open subset of $\mathbb{R}^n$ and F is a closed subset of Ω. Suppose that the vector fields $X(x)$ is Lipschitz in Ω and is tangent to F. Then every integral curve of X which meets F at a point is entirely contained in F.*

Remark 1.4.2. *The proof of Proposition 1.4.1 is similar to the classical proof of the Cauchy-Lipschitz uniqueness theorem for the solutions of ordinary differential equations.*

Proof of Proposition 1.4.1: We shall use the contradictive method. Suppose that these exists an integral curve $x(t)$ satisfying $x'(t) = X(x(t))$, meeting F but not contained in F. We can then find an interval $[t_0, t_1]$ such that

$$x(t_0) = x_0 \in F \quad \text{and} \quad x(t) \notin F \text{ for } t \in]t_0, t_1].$$

Next, we have two claims (here we omit the proofs).

Claim 1: Let $\delta(t)$ be the distance of $x(t)$ to F. There exists a positive constant K such that, for $t \in]t_0, t_1]$, we have

$$\liminf_{h \to 0} \frac{\delta(t+h) - \delta(t)}{|h|} \geq -K\delta(t).$$

Claim 2: Let f be a continuous function on an interval and satisfies, for every t in this interval, that

$$\liminf_{h \to 0} \frac{f(t+h) - f(t)}{|h|} \geq -M \quad \text{with} \quad M > 0,$$

then f is Lipschitz and its Lipschitz constant is M.

Finally, let

$$\theta = \min\big(t_1 - t_0, \frac{1}{2K}\big), \quad \text{and} \quad \varepsilon = \sup \delta(t) \text{ for } t \in [t_0, t_0 + \theta].$$

From the above two claims, the function δ is Lipschitz of constant $K\varepsilon$ in $[t_0, t_0 + \theta]$. Then $\delta(t) \leq \theta K \varepsilon \leq \varepsilon/2$ for $t \in [t_0, t_0 + \theta]$, and this is a contradiction. □

Proposition 1.4.2. *Let $X_1, \ldots, X_m$ be the C^∞ class vectors fields and $Z \in \mathfrak{X}(X_1, \ldots, X_m)$. Then every integral curve of Z can be approached uniformly by piecewise differentiable curves, whose each differentiable arc is an integral curve for one of the vector fields X_i.*

Remark 1.4.3. *To prove Proposition 1.4.2, we need the following lemma. For more details, one can refer to [5].*

Lemma 1.4.1. *Let $x(t)$ be the solution of*

$$\begin{cases} x'(t) = Z(x(t)), \\ x(0) = x_0. \end{cases}$$

On the other hand, let $y(t)$ be a Lipschitz function satisfying almost everywhere that

$$\begin{cases} y'(t) = Z(y(t)) + \omega(t), \\ y(0) = x_0. \end{cases}$$

Then

$$|x(t) - y(t)| \leq \frac{\varepsilon}{M}(e^{Mt} - 1),$$

where $\varepsilon = \sup |\omega(t)|$ and M is the Lipschitz constant of Z.

Proposition 1.4.3. *Let Ω be an open set of $\mathbb{R}^n$ and F a closed subset of Ω. Let $X_1, \dots, X_m$ be the C^∞ class vector fields in Ω, and each of them is tangent to F. On the other hand, assume $Z \in \mathfrak{X}(X_1, \dots, X_m)$. Then, Z is tangent to F, and every integral curve of Z which meets F at a point is entirely contained in F.*

Proof: In fact, Let Γ be an integral curve of Z, passing the point $x_0 \in F$. We can approach it by piecewise differential curves, each arc of which is an integral curve for one of the X_i. From Proposition 1.4.1, these curves are contained in F, i.e. $\Gamma \subset F$. The vector field Z is then necessarily tangent to F. In fact, if there exists a sphere which is outside of F and meeting F only at one point x, and if the normal vector of the sphere at this point is not orthogonal to $Z(x)$, then the integral curve of Z, passing by x, will go through in the sphere and will not be contained in F anymore. □

Proof of Theorem 1.4.1: Theorem 1.4.1 can be deduced by Proposition 1.4.1, Proposition 1.4.2 and Proposition 1.4.3. One can find the details in [5]. □

1.4.2 Linear Case

We consider

$$\begin{cases} -\triangle_X u(x) = f(x), & \text{in } \Omega, \\ u(x) = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.4.1}$$

where Ω is a bounded open domain of $\mathbb{R}^n$, the real vector fields $X = \{X_1, X_2, \cdots, X_m\}$ is C^∞ and satisfies Hörmander's condition on Ω. $\partial\Omega$ is C^∞ smooth and non-characteristic for the system of vector fields X.

Proposition 1.4.4 (Poincaré inequality)**.** *Suppose $\partial\Omega$ is C^∞ and non-characteristic for X, then the first eigenvalue λ_1 of Dirichlet problem for $-\triangle_X$ is positive and we have the following Poincaré inequality*

$$\lambda_1 \|u\|^2_{L^2(\Omega)} \leq \int_\Omega |Xu|^2 dx, \ \forall u \in H^1_{X,0}(\Omega).$$

Proof of Proposition 1.4.4: We set

$$\lambda_1 = \inf_{\|\varphi\|_{L^2(\Omega)}=1, \varphi \in H^1_{X,0}(\Omega)} \{\|X\varphi\|^2_{L^2(\Omega)}\}.$$

Suppose that $\lambda_1 = 0$. Then there exists $\{\varphi_j\} \subset H^1_{X,0}(\Omega)$ such that $\|X\varphi_j\|_{L^2(\Omega)} \to 0$ and $\|X\varphi_j\|_{L^2(\Omega)} = 1$. By the Sobolev embedding (see Theorem 1.3.5 and Remark 1.3.6), $H^1_{X,0}(\Omega)$ is compactly embedded into $L^2(\Omega)$. The variational calculus deduces that there exists $\tilde{\varphi} \in H^1_{X,0}(\Omega)$, $\|\tilde{\varphi}\|_{L^2(\Omega)} = 1$, $\tilde{\varphi} \geq 0$ satisfying

$$\triangle_X \tilde{\varphi} = 0, \ \|X\tilde{\varphi}\|_{L^2(\Omega)} = 0.$$

The Hörmander's theorem of square sum (Theorem 1.1.1) implies that $\triangle_X$ is hypoelliptic in Ω, and then we have $\tilde{\varphi} \in C^\infty(\Omega)$ and

$$X_j\tilde{\varphi}(x) = 0,\ \forall\ x \in \Omega,\ j = 1, \cdots, m.$$

This implies that $\tilde{\varphi}$ is constant along the integral paths of vector fields of $X_1, \cdots, X_m$. Now Rashevski-Chow's connectivity theorem (Theorem 1.2.1) implies that $\tilde{\varphi}$ is constant on each connected component of Ω.

Since $\partial\Omega$ is non-characteristic, by taking $x_0 \in \partial\Omega$, then there exists a X_j such that if $X_j\tilde{\varphi} = 0$ we have $\tilde{\varphi}(x) = 0$ near x_0, which means $\tilde{\varphi}(x) = 0$ on Ω. This is impossible because $\|\tilde{\varphi}\|_{L^2(\Omega)} = 1$, so we prove finally $\lambda_1 > 0$. □

Definition 1.4.1. (1) *The bilinear form $B[\ ,\ ]$ associated with the operator $-\triangle_X$ is*

$$B[u, v] := \int_\Omega \sum_{j=1}^m X_j u X_j v dx,\ \textit{for } u, v \in H^1_{X,0}(\Omega).$$

(2) *We say that $u \in H^1_{X,0}(\Omega)$ is a weak solution of the boundary-value problem* (1.4.1) *if*

$$B[u, v] - \int_\Omega f v dx = 0,\ \forall v \in H^1_{X,0}(\Omega). \tag{1.4.2}$$

Theorem 1.4.2 (Existence)**.** *If $f(x) \in L^2(\Omega)$, then there is a weak solution of* (1.4.1) *$u(x) \in H^{1,2}_{X,0}(\Omega)$.*

Proposition 1.4.5 (Lax-Milgram Theorem, cf. [14])**.** *Assume that*

$$B:\ H \times H \to \mathbb{R}$$

is a bilinear mapping, for which there exist constants $\alpha, \beta > 0$ such that

$$B[u, v] \leq \alpha \|u\|_H \|v\|_H,\quad (u, v \in H)$$

and

$$B[u, u] \geq \beta \|u\|_H^2,\quad (u \in H).$$

Finally, let $f : H \to \mathbb{R}$ be a bounded linear functional on H. Then there exists a unique element $u \in H$ such that

$$B[u, v] =< f, v >,\ \textit{for all } v \in H.$$

Proof of Theorem 1.4.2: First, by direct calculations, we have

$$B[u, v] \leq \|u\|_{H^1_{X,0}(\Omega)} \|v\|_{H^1_{X,0}(\Omega)} \text{ and } B[u, u] = \|u\|^2_{H^1_{X,0}(\Omega)},\ \text{for all } u, v \in H^1_{X,0}(\Omega)$$

Then Proposition 1.4.5 (Lax-Milgram Theorem) implies the results of Theorem 1.4.2. □

Theorem 1.4.3 ($S^{k,\alpha}(\Omega)$ regularity)**.** *If $f \in S^{k,\alpha}(\bar{\Omega})$, $0 \leq \alpha < 1$, $k \in \mathbb{N}$, $u \in H^1_{X,0}(\Omega)$ is a solution of $-\triangle_X u = f$, then $u(x) \in S^{k+2,\alpha}(\bar{\Omega})$.*

Let $u \in C(\Omega)$ be a weak solution of the problem $-\triangle_X u = f$, then $u = u_1 + u_2$ such that

$$\triangle_X u_1 = 0, \text{ in } \Omega, \tag{1.4.3}$$

and

$$u_2(x) = \int_\Omega G(x,y) f(y) dy, \tag{1.4.4}$$

where $G(x,y)$ is the fundamental solution of $-\triangle_X$ (See Proposition 1.3.4). The Hörmander's condition implies that $u_1 \in C^\infty(\Omega)$. Then for any $K \subset\subset \Omega$, and $k \in \mathbb{N}$, there exists a constant $D = D_k$ which depends on K, k, X and $|u_1|_{L^\infty(\Omega)}$ only, such that

$$\|u_1\|_{S^k(K)} \le D_k.$$

Proposition 1.4.6. *Let $f \in S^{k,\alpha}(\Omega)$, with supp $f \in B_1 = B(x_0, R)$, and $u \in C(\Omega)$ be a weak solution of the problem $-\triangle_X u = f$. Then*

$$\|u\|_{S^{k+2,\alpha}(B_1)} \le D_k + C\|f\|_{S^{k,\alpha}(B_1)} \tag{1.4.5}$$

where D_k, C are the constants independent of f.

Proof: It is sufficient to prove that for $f \in S^\alpha(\Omega)$ with supp $f \in B_1 = B(x_0, R)$, then

$$\|u_2\|_{S^{2,\alpha}(B_1)} \le C\|f\|_{S^\alpha(B_1)}. \tag{1.4.6}$$

Step 1: Prove that $u_2 \in S^1(B_1)$ and

$$X_j u_2(x) = \int_{B_1} X_j(x) G(x,y) f(y) dy, j = 1, 2, \cdots, m, x \in B_1, \tag{1.4.7}$$

and $|X_j u_2|_{0,B_1} \le CR|f|_{0,B_1}$.
Step 2: Prove that $u_2 \in S^2(B_1)$ and

$$X_i X_j u_2(x) = \int_{B_1} X_i(x) X_j(x) G(x,y)(f(y) - f(x)) dy + f(x) \int_{B(x_0,2R)} G_0^{ij}(x,y) dy, x \in B_1, \tag{1.4.8}$$

for $j = 1, 2, \cdots, m$, where $G_0^{ij}(x,y) = X_i(x) X_j(x) G(x,y)$ satisfies the estimate (1.3.12).
Step 3: Prove that $u_2 \in S^{2,\alpha}(B_1)$.

In fact, for $x, \bar{x} \in B_1$, set $\delta = \rho(x, \bar{x})$, and take $\xi \in B_1$ such that $\rho(x,\xi), \rho(\bar{x},\xi) \le \delta/2$, then for $i, j = 1, \cdots, m$, we have

$$\begin{aligned} &X_i X_j u_2(x) - X_i X_j u_2(\bar{x}) \\ &\quad = \int_{B(\xi,\delta)} X_i X_j \bar{G}(\bar{x},y)(f(\bar{x}) - f(y)) dy + \int_{B(\xi,\delta)} X_i X_j \bar{G}(\bar{x},y)(f(\bar{x}) - f(y)) dy \\ &\qquad + \int_{B_1 \setminus B(\xi,\delta)} (X_i X_j \bar{G}(\bar{x},y) - X_i X_j \bar{G}(x,y))(f(y) - f(\bar{x})) dy \\ &\qquad + \int_{B_1 \setminus B(\xi,\delta)} X_i X_j \bar{G}(\bar{x},y)(f(\bar{x}) - f(x)) dy + (f(x) - f(\bar{x})) \int_{B(x_0,2R)} G_0^{ij}(x,y) dy \\ &\qquad + f(\bar{x}) \int_{B(x_0,2R)} (G_0^{ij}(x,y) - G_0^{ij}(\bar{x},y)) dy =: I_1 + I_2 + I_3 + I_4 + I_5 + I_6. \end{aligned}$$

where $\bar{G}(x,y) = \phi(x) G(x,y) \phi(y)$ and $\phi \in C_0^\infty(B(x_0, 2R))$ with $\phi(x) = 1$ on B_1 and $|X^J \phi| \le C_J / R^{|J|}$.

Next, we can deduce that (here we omit the proofs)

$$|I_i| \le C\delta^\alpha [f]^X_{\alpha,B_1}, \text{ for } i = 1, \cdots, 6.$$

Then

$$X_iX_ju_2(x) - X_iX_ju_2(\bar{x}) \le C\rho(x,\bar{x})^\alpha [f]^X_{\alpha,B_1},$$

for $x, \bar{x} \in B_1$, with C depending only on α, n. This completes the proof of Proposition 1.4.6. □

Theorem 1.4.4. *Let $f \in S^{k,\alpha}(\Omega)$ for some $k \in \mathbb{N}$, $\alpha > 0$ and $u \in C(\Omega)$ be a weak solution of the equation $-\triangle_X u = f$. Then for all $x_0 \in \Omega$, there exists $R > 0$ such that*

$$\|u\|_{S^{k+2,\alpha}(B_t)} \le C_k\|f\|_{S^{k,\alpha}(B_s)} + \bar{C}_k\big(\|f\|_{L^\infty(B_1)} + \|u\|_{L^\infty(B_1)}\big)\big((s-t)R\big)^{-r(k+\alpha)}, \quad (1.4.9)$$

for all $0 < t < s \le 1$, where C_k and $\bar{C}_k$ are independent of f.

From [56], we have (see [56], Lemma 3.8)

Lemma 1.4.2. *Let $\varphi(t)$ be a non-negative bounded function on $[T_0, T_1]$ with $0 \le T_0 < T_1$. Assume that for any $T_0 \le t < s < T_1$ we have*

$$\varphi(t) \le \theta\varphi(x) + \frac{A}{(s-t)^\beta} + B$$

with $1 > \theta > 0$, $A, B, \beta \ge 0$; then we have

$$\varphi(t) \le C\Big\{\frac{A}{(s-t)^\beta} + B\Big\}$$

for all $T_0 \le t < s \le T_1$, where C depends on β and θ only.

Proof of Theorem 1.4.4: Using (1.2.3) we have, for $0 < t < s \le 1$,

$$d_E(\partial B_s, B_t) \ge C((s-t)R)^Q,$$

where $B_t = B(x_0, tR)$ and d_E the Euclidean distance, thus there exists a function $\zeta \in C_0^\infty(B_s)$ such that $\zeta(x) = 1$ on B_t and

$$[X^k\zeta]_0 + ((s-t)R)^{Q\alpha}[X^k\zeta]^X_\alpha \le C_k((s-t)R)^{-Qk}, \quad (1.4.10)$$

for all $k \in \mathbb{N}$, where $[X^k\zeta] = \sum_{|J|=k}[X^J\zeta]$.

Let $f \in S^{k,\alpha}(\Omega)$ and $u \in C(\Omega)$ be a weak solution of the equation $-\triangle_X u = f$. Then

$$L(\zeta u) = \zeta f - \sum_{j=1}^m 2X_j\zeta X_j u - \sum_{j=1}^m (X_j^2\zeta)u.$$

Using Proposition 1.4.6 and the interpolation inequality (Proposition 1.3.1), we have

$$\begin{aligned}\|\zeta u\|_{S^{k+2,\alpha}(B_s)} \le & D_k + C_k\big\{[X^kf]^X_{\alpha,B_s} + \varepsilon[X^{k+2}u]^X_{\alpha,B_s} \\ & + \|f\|_{S^{k,0}(B_s)}((s-t)R)^{-Q(k+\alpha)} + C|u|_{0,B_s}((s-t)R)^{-Q(k+\alpha)}\big\}.\end{aligned} \quad (1.4.11)$$

Then Lemma 1.4.2 and (1.4.11) imply (1.4.9). □

Proof of Theorem 1.4.3: We prove the result by dividing the problem into following two steps.

Step 1, Interior regularity. Since the problem is local, given $x_0 \in \Omega$, We only need to prove $u(x) \in S^{k+2,\alpha}(B(x_0, R_0))$ for $R_0 > 0$ small enough and $B(x_0, R_0) \subset \Omega$. Thus the interior regularity can be directly deduced from the result of Theorem 1.4.4.

Step 2, Boundary regularity. In case of $x_0 \in \partial\Omega$, similar to the classical Laplacian equation, we need to use some transforms and then consider the special case only in which $x_0 \in \bar{U}$ and $U = \Omega \cap B(x_0, R) = \{x \in B(x_0, R) \mid x_n > \gamma(x_1, \cdots, x_{n-1})\}$, where γ is the definition function of the boundary near x_0. Here the Bony's maximum principle plays a crucial role. We omit the proof here and one can refer to [14] for the more details. $\square$

Similarly, we have

Theorem 1.4.5 ($H_X^{k,p}(\Omega)$ regularity)**.** *If $f \in H_X^{k,p}(\Omega)$, $1 \le p < +\infty$, $k \in \mathbb{N}$, $u \in H_{X,0}^{1,2}(\Omega)$ is a solution of $-\triangle_X u = f$, then $u(x) \in H_X^{k+2,p}(\Omega)$.*

Proof: The detail proof of Theorem 1.4.5 can be found in [51], Theorem 16. $\square$

1.4.3 Nonlinear Case

Here we suppose that the real vector fields $X = \{X_1, X_2, \cdots, X_m\}$ is C^∞ and satisfies Hörmander's condition on a neighborhood of $\bar{\Omega}$. Then we consider

$$\begin{cases} -\triangle_X u(x) = \lambda u + u^q, & \text{in } \Omega, \\ u(x) = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.4.12}$$

where Ω is a bounded open domain of $\mathbb{R}^n$, $1 < q < (\nu+2)/(\nu-2)$, ν is the general Métivier index of X on Ω. $\partial\Omega$ is C^∞ smooth and non-characteristic for X.

Theorem 1.4.6. *Assume λ_1 is the first Dirichlet eigenvalue of $-\triangle_X$, $0 < \lambda < \lambda_1$ and $1 < q < (\nu+2)/(\nu-2)$. Then there exists a non-trivial solution $u \in H_{X,0}^1(\Omega)$ of the problem* (1.4.12).

Proof: We consider the minimization problem

$$i_\lambda = \inf \Big\{ \int_\Omega \sum_{k=1}^m |X_k u(x)|^2 dx - \lambda \int_\Omega |u(x)|^2 dx;\ u \in H_{X,0}^1(\Omega), \|u\|_{L^{q+1}(\Omega)} = 1 \Big\}. \tag{1.4.13}$$

Since $0 < \lambda < \lambda_1$, we have $i_\lambda > 0$. Let $\{u_j\} \subset H_{X,0}^1(\Omega)$ be a minimizing sequence for (1.4.13), i.e., a sequence such that

$$A_\lambda(u_j) = \int_\Omega \sum_{k=1}^m |X_k u_j(x)|^2 dx - \lambda \int_\Omega |u_j(x)|^2 dx \to i_\lambda,$$

and $\|u_j\|_{L^{q+1}(\Omega)} = 1$. Without loss of generality, we can suppose that $u_j \ge 0$ (otherwise we can replace $\{u_j\}$ by $\{|u_j|\}$). Since $\{A_\lambda(u_j)\}$ and $\{\|u_j\|_{L^{q+1}(\Omega)}\}$ are bounded, then $\{u_j\}$ is bounded in $H_{X,0}^1(\Omega)$, and there is a subsequence converging weakly in $H_{X,0}^1(\Omega)$ to $u_0 \in H_{X,0}^1(\Omega)$. By the compactness result of Theorem 1.3.5, the subsequence converges in $L^{q+1}(\Omega)$ norm, so $\|u_0\|_{L^{q+1}(\Omega)} = 1$. By Hölder's inequality, $\lambda \int_\Omega |u_j(x)|^2 dx \to \lambda \int_\Omega |u_0(x)|^2 dx$ and so $A_\lambda(u_0) \le i_\lambda$. But since i_λ is minimum, we necessarily have $A_\lambda(u_0) = i_\lambda$. By a standard variational argument, u_0 satisfies

$$-\triangle_X u_0 = \lambda u_0 + i_\lambda u_0^q,$$

in the distribution sense. We have proved Theorem 1.4.6 for $u = (i_\lambda)^{1/(q-1)} u_0 \in H^1_{X,0}(\Omega)$. $\square$

We study now the regularity of the weak solution in Theorem 1.4.6.

Proposition 1.4.7. *Suppose that $f \in L^s(\Omega)$, $s > \nu/2$, $u \in L^{2\nu/(\nu-2)}(\Omega)$, $u \geq 0$, and*

$$\begin{cases} -\triangle_X u(x) = fu, & \textit{in } \Omega, \\ u(x) = 0, & \textit{on } \partial\Omega. \end{cases}$$

Then we have that u is Hölder continuous in $\bar{\Omega}$, and for some $\beta > 0$, we have $u \in S^\beta(\bar{\Omega})$.

Proof: By Hölder's inequality $fu \in L^{q_0}(\Omega)$ for $1/q_0 = (\nu-2)/(2\nu) + 1/s$. Theorem 1.4.5 implies that $u \in H^{2,q_0}_X(\Omega)$, and thus by Theorem 1.3.5(1),

$$u \in L^{p_1}(\Omega), \text{ for } 1/p_1 = 1/q_0 - 2/\nu = (\nu-2)/(2\nu) - (2/\nu - 1/s).$$

Repeating this argument, we can deduce that

$$u \in L^{p_k}(\Omega), \text{ for } 1/p_k = (\nu-2)/2\nu - k(2/\nu - 1/s) \text{ and } 1/p_k > 0.$$

Suppose k is the largest possible. Then $p_k > \nu/2$ and $u \in H^{2,p_k}_X(\Omega)$, and so Theorem 1.3.5(2) gives $u \in S^\beta(\bar{\Omega})$ for $0 < \beta < 2 - \nu/p_k$. $\square$

Theorem 1.4.7. *Suppose that $f, g \in C^\infty(\Omega)$, $u \in L^{2\nu/(\nu-2)}$, $u \geq 0$ on Ω and*

$$-\triangle_X u = gu + fu^q, \ \textit{in } \Omega,$$

for $2 < q < (\nu+2)/(\nu-2)$. Then $u \in C^\infty(\Omega) \cap S^{2,\beta}(\bar{\Omega})$ for some $0 < \beta < 1$, and $u > 0$ on Ω.

Proof: Let $h = g + fu^{q-1} \in L^{2\nu/((\nu-2)(q-1))}(\Omega)$, then $s = 2\nu/((\nu-2)(q-1)) > \nu/2$. It follows from Proposition 1.4.7 that $u \in S^\beta(\bar{\Omega})$ for some $0 < \beta < 1$ and $u > 0$ on Ω. Since $q > 2$ and u is bounded away from zero, we also have $u^q \in S^\beta(\bar{\Omega})$. Thus, we conclude from Theorem 1.4.3 that $u \in S^{2,\beta}(\bar{\Omega})$. From the Bony's maximum principle we have $u > 0$ on Ω. Then in the interior of Ω, $u^q \in S^{2,\beta}(\bar{\Omega})$, so we can repeat this argument in the interior of Ω, and by induction we can deduce that $u \in C^\infty(\Omega)$, which proves the result of Theorem 1.4.7. If $q \in \mathbb{N}$, we can also obtain $u \in C^\infty(\bar{\Omega})$. $\square$

Next, we consider

$$\begin{cases} -\triangle_X u(x) + a(x)u = u^q, & \text{in } \Omega, \\ u(x) = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.4.14}$$

where Ω is a bounded open domain of $\mathbb{R}^n$, $q = (\nu+2)/(\nu-2)$ is the critical Sobolev embedded exponent, ν is the general Métivier index of X on Ω.

For $u \in H^1_{X,0}(\Omega)$, let

$$I = \inf\{\|Xu\|^2_{L^2(\Omega)} + \int_\Omega a(x)u^2(x)dx; u \in H^1_{X,0}(\Omega), \ \int_\Omega |u|^{2\nu/(\nu-2)}dx = 1\},$$

and

$$S = \inf\{\int_\Omega |Xu|^2 dx; u \in H^1_{X,0}(\Omega), \ \int_\Omega |u|^{2\nu/(\nu-2)}dx = 1\}.$$

Theorem 1.4.8 (Concentration-compactness principle on C-C space). *Let $\{u_j\}$ be a bounded sequence and converges to u weakly in $H^1_{X,0}(\Omega)$ and $|Xu_j|^2dx \rightharpoonup \mu$, $|u_j|^{2\nu/(\nu-2)}dx \rightharpoonup \eta$ weakly in the sense of measure where μ and η are bounded non-negative measures. Then*

(1) *There exists at most a countable set J, a family $\{x_j\}_{j\in J}\subset\Omega$ and $\{\eta_j, j\in J\}$ of positive numbers such that*

$$\eta = |u|^{2\nu/(\nu-2)}dx + \sum_{j\in J}\eta_j\delta_{x_j}.$$

(2) *In addition we have*

$$\mu = |Xu|^2dx + S\sum_{j\in J}\eta_j^{(\nu-2)/\nu}\delta_{x_j} \text{ and } \sum_{j\in J}\eta_j^{(\nu-2)/\nu} < \infty.$$

We consider that there exists $\alpha > 0$ such that

$$\|X\varphi\|^2_{L^2(\Omega)} + \int_\Omega a(x)\varphi^2(x)dx \geq \alpha\|X\varphi\|^2_{L^2(\Omega)}. \tag{1.4.15}$$

Theorem 1.4.9. *Suppose that $a(x)\in C^\infty(\bar\Omega)$ satisfies* (1.4.15) *and $\{u_j\}$ is the minimizing sequence of I. If $I < S$, then $\{u_j\}$ is a relative compactness of minimizing sequence for I in $H^1_{X,0}(\Omega)$. Hence there is a minimal element $u\in H^1_{X,0}(\Omega)$. If $I > 0$, then there exists a constant C such that Cu is the weak solution of* (1.4.14)*. Moreover $u\in C^\infty(\Omega)\cap C^\alpha(\bar\Omega)$, for some $\alpha > 0$.*

Remark 1.4.4. *The detail proofs of Theorem 1.4.8 and Theorem 1.4.9 can be found in [2], in which the technique of micro-local analysis has been used.*

1.5 Estimates of Eigenvalues in Finitely Degenerate Cases

1.5.1 Retrospect: the Classical Cases

Let us consider the following Dirichlet eigenvalue problems in $H^1_{X,0}(\Omega)$,

$$\begin{cases} -\triangle_X u = \lambda u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.5.1}$$

In the classical case, $X = \{\partial_{x_1}, \cdots, \partial_{x_n}\}$, $\triangle_X$ is the Laplacian $\triangle$.

Proposition 1.5.1 (Weyl's asymptotic formula, cf. [55]). *The k-th Dirichlet eigenvalue for $-\triangle$ satisfies*

$$\lambda_k \sim C_n(k/|\Omega|_n)^{2/n}, \tag{1.5.2}$$

where $|\Omega|_n$ is the n-dimensional Lebesgue measure of Ω and $C_n = (2\pi)^2B_n^{-2/n}$ with B_n being the volume of the unit ball in $\mathbb{R}^n$.

Remark 1.5.1. *Pólya [47] proved that the asymptotic relation* (1.5.2) *is in fact a one-sided inequality if Ω is a plane domain which tiles $\mathbb{R}^2$ (and his proof also works in $\mathbb{R}^n$). Also he proposed following conjecture which is still open.*

Pólya Conjecture: *the inequality*

$$\lambda_k \geq C_n(k/|\Omega|_n)^{2/n}, \text{ for any } k\geq 1, \tag{1.5.3}$$

holds for any domain Ω in $\mathbb{R}^n$.

Proposition 1.5.2 (Li-Yau's inequality, c.f. [36])**.** *The eigenvalues for $-\triangle$ satisfy*

$$\sum_{i=1}^{k}\lambda_i \geq \frac{nC_n}{n+2}k^{\frac{n+2}{n}}|\Omega|_n^{-\frac{2}{n}},\ \textit{for any } k\geq 1, \tag{1.5.4}$$

where $|\Omega|_n$ is the n-dimensional Lebesgue measure of Ω and $C_n=(2\pi)^2B_n^{-2/n}$ with B_n being the volume of the unit ball in $\mathbb{R}^n$.

For the upper bounds of eigenvalues, Payne et al. [46] proved

$$\lambda_{k+1}-\lambda_k \leq \frac{4}{nk}\sum_{i=1}^{k}\lambda_i.$$

Further, in 1991, Yang [57] proved a very sharp universal inequality:

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \leq \frac{4}{n}\sum_{i=1}^{k}\lambda_i(\lambda_{k+1}-\lambda_i).$$

In 2007, from the above Yang's inequality and recursion formula, Cheng and Yang [11] proved

$$\lambda_{k+1}\leq k^{\frac{2}{n}}\lambda_1,\ \text{for large } k \text{ and } n.$$

On the other hand, if there exists a constant c_0 such that

$$r|\Omega_r|_n \leq c_0|\Omega|_n^{(n-1)/n},\ \text{for every } r>|\Omega|_n^{-1/n}, \tag{1.5.5}$$

where $\Omega_r=\{x\in\Omega \mid dist(x,\partial\Omega)<r\}$. Then Kröger [34] gained that

$$\sum_{i=1}^{k}\lambda_i \leq \frac{nC_n}{n+2}k^{\frac{n+2}{n}}|\Omega|_n^{-\frac{2}{n}}+\bar{C}_nk^{\frac{n+1}{n}}, \tag{1.5.6}$$

for any $k\geq c_0^n$, where $\bar{C}_n$ is a constant which depends only on Ω and n.

1.5.2 Asymptotic Estimates and Lower Bounds

Proposition 1.5.3. *Suppose the system of vector fields X satisfies Hörmander's condition on a neighborhood of Ω. If $\partial\Omega$ is C^∞ and non-characteristic for X, then the operator $-\triangle_X$ has a sequence of discrete Dirichlet eigenvalues $0<\lambda_1\leq\lambda_2\leq\lambda_3\leq\cdots\leq\lambda_k\leq\cdots$, and $\lambda_k\to\infty$, such that for any $k\geq 1$, the Dirichlet problem*

$$\begin{cases}-\triangle_X\varphi_k=\lambda_k\varphi_k, & \textit{in } \Omega,\\ \varphi_k=0, & \textit{on } \partial\Omega,\end{cases}$$

admits a non trivial solution $\varphi_k\in H^1_{X,0}(\Omega)$. Moreover, $\{\varphi_k\}_{k\geq1}$ constitute an orthonormal basis of the Sobolev space $H^1_{X,0}(\Omega)$.

Proof: First, from Proposition 1.4.4, it holds that

$$(-\triangle_Xu,u)_{L^2(\Omega)}=\|Xu\|^2_{L^2(\Omega)}\geq\lambda_1\|u\|^2_{L^2(\Omega)},\quad \forall\ u\in H^1_{X,0}(\Omega),\ \text{and } u\not\equiv0.$$

Also
$$(-\triangle_X u, v)_{L^2(\Omega)} = (u, -\triangle_X v)_{L^2(\Omega)}, \quad \forall u,\ v \in H^1_{X,0}(\Omega).$$
Then the operator $-\triangle_X$ is positive definite and self-adjoint on $H^1_{X,0}(\Omega)$. Lax-Milgram Theorem implies that for any $g \in H_X^{-1}(\Omega)$, the following Dirichlet problem

$$\begin{cases} -\triangle_X u = g, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

admits a unique solution $u \in H^1_{X,0}(\Omega)$, where $H_X^{-1}(\Omega)$ is the dual space of $H^1_{X,0}(\Omega)$ with the norm
$$\|g\|_{H_X^{-1}(\Omega)} = \sup_{\phi \in C_0^\infty, \phi \neq 0} \frac{|\langle g, \phi\rangle|}{\|\phi\|_{H^1_{X,0}(\Omega)}},$$
and $-\triangle_X : H^1_{X,0}(\Omega) \to H_X^{-1}(\Omega)$ is continuous. Thus the inverse operator $-\triangle_X^{-1}$ is well-defined and is a continuous map from $H_X^{-1}(\Omega)$ to $H^1_{X,0}(\Omega)$. The compact embedding i: $H^1_{X,0}(\Omega) \to L^2(\Omega)$ and the continuous embedding i^*: $L^2(\Omega) \to H_X^{-1}(\Omega)$ imply that

$$K := -\triangle_X^{-1} \circ i^* \circ i : H^1_{X,0}(\Omega) \to H^1_{X,0}(\Omega)$$

is compact and self-adjoint. Then there exist eigenvalues $\{\eta_k\}$ of compact operator K such that $\eta_k > 0$, for $k \geq 1$ and $\eta_k \to 0$. If $\{\phi_k\}$ are the associated normal eigenfunctions, we have that $K\phi_k = \eta_k \phi_k$ for any $k \geq 1$ and $\{\phi_k\}$ form a complete basis of Hilbert space $H^1_{X,0}(\Omega)$. This completes the proof. □

Proposition 1.5.4 (Métivier's asymptotic formula, cf. [37]). *If X satisfies Hörmander's condition and Métivier's condition on a neighborhood of Ω, then the following asymptotic result*
$$\lambda_k \approx k^{\frac{2}{\nu}}, \ \text{as } k \to +\infty, \tag{1.5.7}$$
holds, where Métivier index ν is defined by (1.2.4)*.*

For general finitely degenerate operator, by using the sub-elliptic estimate (see Theorem 1.1.2), we can deduce that

Theorem 1.5.1. *Suppose the system of vector fields X satisfies the Hörmander's condition on Ω with the Hörmander index Q. Let λ_j be the j^{th} Dirichlet eigenvalue of the problem* (1.5.1)*, then for all $k \geq 1$,*
$$\sum_{j=1}^{k} \lambda_j \geq C_1 k^{1+\frac{2}{Qn}} - \widetilde{C}(Q)k, \tag{1.5.8}$$
where $C_1 = \frac{nQ(2\pi)^{\frac{2}{Q}}}{C(Q)\cdot(nQ+2)(|\Omega|_n B_n)^{\frac{2}{nQ}}}$, $C(Q)$ and $\widetilde{C}(Q)$ are the constants in Theorem 1.1.2, B_n is the volume of the unit ball in $\mathbb{R}^n$, $|\Omega|_n$ is the volume of Ω.

Remark 1.5.2. (1) *Since $k\lambda_k \geq \sum_{j=1}^k \lambda_j$, then Theorem 1.5.1 show that the Dirichlet eigenvalues λ_k satisfy*
$$\lambda_k \geq C_1 k^{\frac{2}{Qn}} - \widetilde{C}(Q), \ \text{for all } k \geq 1.$$

(2) *If $\triangle_X = \triangle$ is Laplacian, then the Hörmander index $Q = 1$, $C(Q) = 1$ and $\widetilde{C}(Q) = 0$. Thus for all $k \geq 1$, the lower bound estimate* (1.5.8) *gives the same result to the Li-Yau's estimate* (1.5.4)*.*

(3) However, when Hörmander index $Q > 1$, the increasing order of k in the lower bounds (1.5.8) is $2/(Qn)$, which is smaller than the order of k in the Métivier's asymptotic formula (1.5.7). That means the lower bounds of Dirichlet eigenvalues in (1.5.8) are not precise. Indeed, one example below with $Q = 2$ gives a precise lower bounds of Dirichlet eigenvalues.

Example 1.5.1. *For the Kohn Laplacian in Heisenberg group on a bounded $\Omega \subset \mathbb{R}^{2N+1}$, we know that for this example the Hörmander's condition and Métivier's condition are all satisfied with $Q = 2$ and $\nu = 2N + 2$. Then, Hansson and Laptev [20] and [21] proved that*

$$\lambda_k \geq \Big(\frac{(2\pi)^{N+1}(N+1)^{N+2}}{2\bar{C}_N(N+2)^{N+1}|\Omega|}\Big)^{\frac{1}{N+1}} k^{\frac{1}{N+1}}, \text{ for all } k \geq 1,$$

where $\bar{C}_N = \sum_{n_1,\cdots,n_N \geq 0} \frac{1}{(2(n_1+\cdots+n_N)+N)^{N+1}}$.

Now, let us give the lower bounds of the Dirichlet eigenvalues for another class of finitely degenerate elliptic operator which are more precise than the estimates (1.5.8).

Theorem 1.5.2. *Let $X = (\partial_{x_1}, \cdots, \partial_{x_{n-1}}, x_1^l \partial_{x_n})$, $l \in \mathbb{N}$, $n \geq 2$, Ω is a smooth bounded open domain in $\mathbb{R}^n$ and $\Omega \cap \{x_1 = 0\} \neq \emptyset$. Then X satisfies the Hörmander's condition with the Hörmander index $Q = l + 1$. Also the generalized Métivier index $\bar{\nu} = Q + n - 1$. Suppose λ_j be the j^{th} Dirichlet eigenvalue of the problem (1.5.1), then*

$$\sum_{j=1}^{k} \lambda_j \geq C(n, Q, \Omega) k^{1+\frac{2}{Q+n-1}} - \widetilde{C}(Q)k, \text{ for all } k \geq 1,$$

where

$$C(n, Q, \Omega) = \frac{A_Q}{\hat{C}(Q)n(n+Q+1)}\Big(\frac{(2\pi)^n}{|\Omega|_n \omega_{n-1} Q}\Big)^{\frac{2}{n+Q-1}} (n+Q-1)^{\frac{n+Q+1}{n+Q-1}},$$

and

$$\hat{C}(Q) = C(Q) + \min\{1, Q-1\} > 0, \quad A_Q = \begin{cases} \min\{1, n^{\frac{3-Q}{2}}\}, & Q \geq 2, \\ n, & Q = 1; \end{cases}$$

$C(Q)$ and $\widetilde{C}(Q)$ are the constants in Theorem 1.1.2, ω_{n-1} is the area of the unit sphere in $\mathbb{R}^n$, $|\Omega|_n$ is the volume of Ω.

Remark 1.5.3. (1) *$X = (\partial_{x_1}, \cdots, \partial_{x_{n-1}}, x_1^l \partial_{x_n})$ in Theorem 1.5.2 does not satisfy the Métivier's condition.*

(2) *If $l = 0$, then we have*

$$\triangle_X = \triangle, \; Q = 1, \; \hat{C}(Q) = 1, \; C_2 = 0, \; A_Q = n,$$

and $C(n, Q, \Omega) = \frac{n}{n+2}(2\pi)^2 B_n^{-\frac{2}{n}} |\Omega|_n^{\frac{2}{n}}$. Thus the result of Theorem 1.5.2 is the same to the result of Li-Yau's estimate (1.5.4).

The proof of Theorem 1.5.2 is dependent on the following results.

Lemma 1.5.1. *For the system of vector fields $X = (X_1, \cdots, X_m)$, if $\{\psi_j\}_{j=1}^k$ are the set of orthonormal eigenfunctions corresponding to the Dirichlet eigenvalues $\{\lambda_j\}_{j=1}^k$. Define*

$$\Psi(x, y) = \sum_{j=1}^{k} \psi_j(x)\psi_j(y).$$

Then for the partial Fourier transformation of $\Psi(x,y)$ *in the* x*-variable,*

$$\hat{\Psi}(z,y) = (2\pi)^{-n/2}\int_{\mathbb{R}^n}\Psi(x,y)e^{-ix\cdot z}dx,$$

we have

$$\int_{\Omega}\int_{\mathbb{R}^n}|\hat{\Psi}(z,y)|^2dzdy = k, \ \ and \ \int_{\Omega}|\hat{\Psi}(z,y)|^2dy \le (2\pi)^{-n}|\Omega|_n.$$

Lemma 1.5.2. *Let* f *be a real-valued function defined on* $\mathbb{R}^n$ *with* $0 \le f \le M_1$*, and if for* $Q \in \mathbb{Z}^+$*,*

$$\int_{\mathbb{R}^n}(\sum_{i=1}^{n-1} z_i^2 + |z_n|^{2/Q})f(z)dz \le M_2.$$

Then

$$\int_{\mathbb{R}^n} f(z)dz \le \frac{1}{n+Q-1}(M_1Q\omega_{n-1})^{\frac{2}{n+Q+1}}\Big(\frac{n(Q+n+1)}{A_Q}\Big)^{\frac{n+Q-1}{n+Q+1}}M_2^{\frac{n+Q-1}{n+Q+1}},$$

where ω_{n-1} *is the area of the unit sphere in* $\mathbb{R}^n$*, and*

$$A_Q = \begin{cases} \min\{1, n^{\frac{3-Q}{2}}\}, & Q \ge 2, \\ n, & Q = 1. \end{cases}$$

Proposition 1.5.5. *If* X *belongs to the system of vector fields in Theorem 1.5.2, then we have the following sub-elliptic estimate*

$$\sum_{i=1}^{n-1}\big\|\partial_{x_i}u\big\|^2_{L^2(\Omega)} + \big\||\partial_{x_n}|^{1/Q}u\big\|^2_{L^2(\Omega)} \le \hat{C}(Q)(\|Xu\|^2_{L^2(\Omega)} + \widetilde{C}(Q)\|u\|^2_{L^2(\Omega)}), \tag{1.5.9}$$

for all $u \in C_0^\infty(\Omega)$*. Where* $|\partial_{x_n}|^{1/Q}$ *is a pesudo-differential operator with the symbol* $|\xi_n|^{1/Q}$*,* $\hat{C}(Q) = C(Q) + \min\{1, Q-1\} > 0$*,* $C(Q)$ *and* $\widetilde{C}(Q)$ *are the constants in* (1.1.15)*.*

Proof of Theorem 1.5.2: Let $X = (\partial_{x_1}, \cdots, \partial_{x_{n-1}}, x_1^l\partial_{x_n})$, $\{\lambda_k\}_{k\ge1}$ be a sequence of the Dirichlet eigenvalues for the problem (1.5.1), $\{\psi_k(x)\}_{k\ge1}$ be the corresponding eigenfunctions, then $\{\psi_k(x)\}_{k\ge1}$ constitute an orthonormal basis of the Sobolev space $H^1_{X,0}(\Omega)$.

Let $\Psi(x,y) = \sum_{j=1}^k \psi_j(x)\psi_j(y)$. By using Plancherel's formula and Proposition 1.5.5, we have

$$\begin{aligned} &\int_{\mathbb{R}^n}\int_{\Omega}(\sum_{i=1}^{n-1} z_i^2 + |z_n|^{2/Q})|\hat{\Psi}(z,y)|^2dydz \\ &= \int_{\mathbb{R}^n}\int_{\Omega}(\sum_{i=1}^{n-1}\big|\partial_{x_i}\Psi(x,y)\big|^2 + \big||\partial_{x_n}|^{1/Q}\Psi(x,y)\big|^2)dydx \\ &= \int_{\Omega}\int_{\Omega}(\sum_{i=1}^{n-1}\big|\partial_{x_i}\Psi(x,y)\big|^2 + \big||\partial_{x_n}|^{1/Q}\Psi(x,y)\big|^2)dydx \\ &\le \hat{C}(Q)\Big(\int_{\Omega}\int_{\Omega}|X(x)\Psi(x,y)|^2dxdy + \widetilde{C}(Q)\int_{\Omega}\int_{\Omega}|\Psi(x,y)|^2dxdy\Big). \end{aligned} \tag{1.5.10}$$

Next, we can deduce that

$$\begin{aligned}\int_\Omega\int_\Omega |X(x)\Psi(x,y)|^2dxdy &= \int_\Omega\Big(\sum_{l=1}^n\int_\Omega|\sum_{j=1}^k(X_l(x)\psi_j(x))\psi_j(y)|^2dx\Big)dy\\ &= \sum_{l=1}^n\Big(\int_\Omega\sum_{j=1}^k|X_l(x)\psi_j(x)|^2\Big)dx \qquad (1.5.11)\\ &= -\int_\Omega\sum_{j=1}^k\psi_j(x)\triangle_X\psi_j(x)dx=\sum_{j=1}^k\lambda_j.\end{aligned}$$

Thus from Lemma 1.5.1, (1.5.10) and (1.5.11) give that

$$\int_{\mathbb{R}^n}\int_\Omega(\sum_{i=1}^{n-1}z_i^2+|z_n|^{2/Q})|\hat\Psi(z,y)|^2dydz\le\hat C(Q)(\sum_{j=1}^k\lambda_j+\widetilde C(Q)k).$$

Now we choose

$$f(z)=\int_\Omega|\hat\Psi(z,y)|^2dy, M_1=(2\pi)^{-n}|\Omega|_n, M_2=\hat C(Q)\Big(\sum_{j=1}^k\lambda_j+\widetilde C(Q)k\Big).$$

Then the results of Lemma 1.5.1 and Lemma 1.5.2 give that, for any $k\ge 1$,

$$k\le\frac{1}{n+Q-1}\Big(\frac{Q|\Omega|_n\omega_{n-1}}{(2\pi)^n}\Big)^{\frac{2}{n+Q+1}}\Big(\frac{n(Q+n+1)}{A_Q}\Big)^{\frac{n+Q-1}{n+Q+1}}\cdot\Big(\hat C(Q)(\sum_{j=1}^k\lambda_j+\widetilde C(Q)k)\Big)^{\frac{n+Q-1}{n+Q+1}}.$$

This means, for any $k\ge1$,

$$\sum_{j=1}^k\lambda_j\ge C(n,Q,\Omega)k^{1+\frac{2}{n+Q-1}}-\widetilde C(Q)k,$$

with

$$C(n,Q,\Omega)=\frac{A_Q}{\hat C(Q)n(n+Q+1)}\Big(\frac{(2\pi)^n}{|\Omega|_n\omega_{n-1}Q}\Big)^{\frac{2}{n+Q-1}}(n+Q-1)^{\frac{n+Q+1}{n+Q-1}},$$

and

$$\hat C(Q)=C(Q)+\min\{1,Q-1\}>0,\quad A_Q=\begin{cases}\min\{1,n^{\frac{3-Q}{2}}\}, & Q\ge2,\\ n, & Q=1.\end{cases}$$

□

Proof of Lemma 1.5.1: Since

$$\int_{\mathbb{R}^n}\Psi^2(x,y)dx=\int_{\mathbb{R}^n}|\hat\Psi(z,y)|^2dz.$$

Hence by the orthonormality of $\{\psi_j\}_{j=1}^k$, one has

$$\int_\Omega\int_{\mathbb{R}^n}|\hat\Psi(z,y)|^2dzdy=\int_\Omega\int_{\mathbb{R}^n}|\Psi(x,y)|^2dxdy=\int_\Omega\int_\Omega|\Psi(x,y)|^2dxdy=k.$$

On the other hand,

$$\int_{\mathbb{R}^n} |\hat{\Psi}(z,y)|^2 dy = \int_\Omega (2\pi)^{-n} |\int_{\mathbb{R}^n} \Psi(x,y) e^{-ix\cdot z} dx|^2 dy = \int_\Omega (2\pi)^{-n} |\int_\Omega \Psi(x,y) e^{-ix\cdot z} dx|^2 dy.$$

Using the Fourier expansion for the function $e^{-ix\cdot z}$, i.e.

$$e^{-ix\cdot z} = \sum_{j=1}^\infty a_j(z)\psi_j(x), \text{ with } a_j(z) = \int_\Omega e^{-ix\cdot z}\psi_j(x)dx.$$

Then we know that

$$\sum_{j=1}^\infty |a_j(z)|^2 = \int_\Omega |e^{-ix\cdot z}|^2 dx = |\Omega|_n.$$

Thus

$$|\int_\Omega \Psi(x,y)e^{-ix\cdot z}dx| \le |\int_\Omega \sum_{j=1}^k \sum_{l=1}^\infty a_l(z)\psi_l(x)\psi_j(x)\psi_j(y)dx| = |\sum_{j=1}^k a_j(z)\psi_j(y)|.$$

Using the estimates above, we have

$$\int_\Omega |\hat{\Psi}(z,y)|^2 dy \le (2\pi)^{-n} \int_\Omega |\sum_{j=1}^k a_j(z)\psi_j(y)|^2 dy = (2\pi)^{-n} \sum_{j=1}^k |a_j(z)|^2 \le (2\pi)^{-n}|\Omega|_n.$$

$\square$

Proof of Lemma 1.5.2: First, we choose R such that

$$\int_{\mathbb{R}^n} (\sum_{i=1}^{n-1} z_i^2 + |z_n|^{2/Q})g(z)dz = M_2,$$

where

$$g(z) = \begin{cases} M_1, & \sum_{i=1}^{n-1} z_i^2 + |z_n|^{2/Q} < R^2, \\ 0, & \sum_{i=1}^{n-1} z_i^2 + |z_n|^{2/Q} \ge R^2. \end{cases}$$

Then $(\sum_{i=1}^{n-1} z_i^2 + |z_n|^{2/Q} - R^2)(f(z) - g(z)) \ge 0$, hence

$$R^2 \int_{\mathbb{R}^n} (f(z) - g(z))dz \le \int_{\mathbb{R}^n} (\sum_{i=1}^{n-1} z_i^2 + |z_n|^{2/Q})(f(z) - g(z))dz \le 0. \tag{1.5.12}$$

Now we have

$$\begin{aligned} M_2 &= \int_{\mathbb{R}^n} (\sum_{i=1}^{n-1} z_i^2 + |z_n|^{2/Q})g(z)dz = M_1 \int_{\widetilde{B}_R} (\sum_{i=1}^{n-1} z_i^2 + |z_n|^{2/Q})dz \\ &= M_1 Q \int_{B_R} |z|^2 |z_n|^{Q-1} dz = \frac{M_1 Q}{n} \int_{B_R} |z|^2 (\sum_{i=1}^n |z_i|^{Q-1})dz, \end{aligned} \tag{1.5.13}$$

where

$$\widetilde{B}_R = \{z \in \mathbb{R}^n, \sum_{i=1}^{n-1} z_i^2 + |z_n|^{2/Q} \leq R^2\},\ B_R = \{z \in \mathbb{R}^n, |z| \leq R\}.$$

On the other hand,

$$\sum_{i=1}^{n} |z_i|^{Q-1} = |z|^{Q-1} \sum_{i=1}^{n} (\frac{|z_i|}{|z|})^{Q-1} \geq A_Q |z|^{Q-1}, \tag{1.5.14}$$

where

$$A_Q = \begin{cases} \min\{1, n^{\frac{3-Q}{2}}\}, & Q \geq 2, \\ n, & Q = 1. \end{cases}$$

Then (1.5.13) and (1.5.14) imply

$$M_2 \geq \frac{M_1 Q A_Q}{n} \int_{B_R} |z|^{Q+1} dz = \frac{M_1 Q A_Q \omega_{n-1}}{n(n+Q+1)} R^{n+Q+1}. \tag{1.5.15}$$

From the definition of $g(z)$, we know

$$\begin{aligned} \int_{\mathbb{R}^n} g(z) dz &= M_1 \int_{\widetilde{B}_R} dz = M_1 Q \int_{B_R} |z_n|^{Q-1} dz \\ &\leq M_1 Q \int_{B_R} |z|^{Q-1} dz = \frac{M_1 Q \omega_{n-1}}{n+Q-1} R^{n+Q-1}. \end{aligned} \tag{1.5.16}$$

Combining (1.5.12), (1.5.15) and (1.5.16), we can gain

$$\int_{\mathbb{R}^n} f(z) dz \leq \int_{\mathbb{R}^n} g(z) dz \leq \frac{1}{n+Q-1} (M_1 Q \omega_{n-1})^{\frac{2}{n+Q+1}} \Big(\frac{n(Q+n+1)}{A_Q}\Big)^{\frac{n+Q-1}{n+Q+1}} M_2^{\frac{n+Q-1}{n+Q+1}}.$$

$\square$

Proof of Proposition 1.5.5: First, when $Q = 1$, $\triangle_X = \triangle$ and (1.5.9) is an obvious result. For $Q > 1$ and $u \in C_0^\infty(\Omega)$, from Plancherel's formula, we have

$$\begin{aligned} \big\||\partial_{x_n}|^{1/Q} u\big\|^2_{L^2(\Omega)} &= \big\||\partial_{x_n}|^{1/Q} u\big\|^2_{L^2(\mathbb{R}^n)} = \big\||\xi_n|^{1/Q} \hat{u}\big\|^2_{L^2(\mathbb{R}^n)} \\ &\leq \big\||\xi|^{1/Q} \hat{u}\big\|^2_{L^2(\mathbb{R}^n)} = \big\||\nabla|^{1/Q} u\big\|^2_{L^2(\mathbb{R}^n)} \\ &= \big\||\nabla|^{1/Q} u\big\|^2_{L^2(\Omega)}. \end{aligned} \tag{1.5.17}$$

Also,

$$\sum_{i=1}^{n-1} \big\|\partial_{x_i} u\big\|^2_{L^2(\Omega)} \leq \|Xu\|^2_{L^2(\Omega)}. \tag{1.5.18}$$

Combining (1.1.15), (1.5.17) and (1.5.18), we can gain the sub-elliptic estimate (1.5.9). $\square$

Similarly, we have

Theorem 1.5.3. *Let $X = (\partial_{x_1}, \cdots, \partial_{x_{n-2}}, x_i^p \partial_{x_{n-1}}, x_j^q \partial_{x_n})$, $n \geq 3$, $i, j \in \{1, 2, \cdots, n-2\}$, $p, q \in \mathbb{N}$. If Ω is a smooth bounded open domain in $\mathbb{R}^n$ with $\Omega \cap \{x_i = 0\} \neq \emptyset$ and $\Omega \cap \{x_j = 0\} \neq \emptyset$. Then X satisfies the Hörmander's condition on Ω with $Q = \max\{p, q\} + 1$*

and the generalized Métivier index $\bar{\nu} = n+p+q$. *Suppose* λ_j *be the* j^{th} *Dirichlet eigenvalue of the problem* (1.5.1), *then*

$$\sum_{j=1}^{k} \lambda_j \geq C_1(n,p,q,\Omega)k^{1+\frac{2}{\bar{\nu}}} - C_2(p,q)k, \text{ for all } k \geq 1,$$

where constants $C_1(n,p,q,\Omega) > 0$ *and* $C_2(p,q) = \max\{\tilde{C}(p+1), \tilde{C}(q+1)\} \geq 0$ *are independent of* k, *and* $\tilde{C}(p+1)$, $\tilde{C}(q+1)$ *are constants in* (1.1.15).

To prove Theorem 1.5.3, we need following lemmas.

Lemma 1.5.3. *Let* $X = (\partial_{x_1}, \cdots, \partial_{x_{n-2}}, x_i^p \partial_{x_{n-1}}, x_j^q \partial_{x_n})$, $n \geq 3$, $i,j \in \{1,2,\cdots,n-2\}$, $p,q \in \mathbb{N}$. *If* Ω *is a smooth bounded open domain in* $\mathbb{R}^n$ *with* $\Omega \cap \{x_i = 0\} \neq \emptyset$ *and* $\Omega \cap \{x_j = 0\} \neq \emptyset$. *Then we have the following sub-elliptic estimate*

$$\sum_{l=1}^{n-2} \left\|\partial_{x_l} u\right\|^2_{L^2(\Omega)} + \left\||\partial_{x_{n-1}}|^{\frac{1}{p+1}} u\right\|^2_{L^2(\Omega)} + \left\||\partial_{x_n}|^{\frac{1}{q+1}} u\right\|^2_{L^2(\Omega)} \leq C_1(\|Xu\|^2_{L^2(\Omega)} + C_2\|u\|^2_{L^2(\Omega)}),$$

for all $u \in C_0^\infty(\Omega)$, *where the constants* C_1, C_2 *are only dependent on* p,q,n,Ω.

Lemma 1.5.4. *Let* f *be a real-valued function defined on* $\mathbb{R}^n$ *with* $0 \leq f \leq M_1$. *If*

$$\int_{\mathbb{R}^n} (\sum_{i=1}^{n-2} z_i^2 + |z_{n-1}|^{\frac{2}{p+1}} + |z_n|^{\frac{2}{q+1}}) f(z) dz \leq M_2,$$

with $p,q \in N^+$. *Then*

$$\int_{\mathbb{R}^n} f(z)dz \leq \frac{(p+1)(q+1)\omega_{n-1}}{n+p+q} M_1^{\frac{2}{n+p+q+2}} (\frac{3n^{\frac{n+p+q+2}{2}}}{2^n})^{\frac{n+p+q}{n+p+q+2}} M_2^{\frac{n+p+q}{n+p+q+2}},$$

where ω_{n-1} *is the area of the unit sphere in* $\mathbb{R}^n$.

Remark 1.5.4. *The proof of Theorem 1.5.3, Lemma 1.5.3 and Lemma 1.5.4 are similar to those in Theorem 1.5.2, Lemma 1.5.2 and Proposition 1.5.5.*

Remark 1.5.5. *The result of Theorem 1.5.3 can be deduced to the more general Grushin type degenerate vector fields*

$$X = \{\partial_{x_1}, \cdots \partial_{x_{n-k}}, f_1(\bar{x})\partial_{x_{n-k+1}}, \cdots, f_k(\bar{x})\partial_{x_n}\},$$

where $\bar{x} = (x_1, \cdots, x_{n-k})$ *for* $2 \leq k < n$, $f_j(\bar{x})(1 \leq j \leq k)$ *are smooth functions with finite order zero point in* Ω. *In this case, we can also obtain that the lower bounds of* λ_k *will be at least polynomial increasing in* k *with the power* $2/\bar{\nu}$.

Chapter 2

Infinitely Degenerate Elliptic Equations

2.1 Hypoellipticity and Logarithmic Regularity Estimate

2.1.1 Motivations of Infinitely Degenerate Elliptic Equations from Complex Geometry

Definition 2.1.1 (Infinitely degenerate elliptic operator)**.** *If the system of vector fields X does not satisfy the Hörmander's condition on Ω, then we say that X is an infinitely degenerate system of vector fields on Ω and $\triangle_X = \sum_{i=1}^m X_i^2$ is an infinitely degenerate elliptic operator.*

Example 2.1.1. *Let $X = \{\partial_{x_1}, \partial_{x_2}, \cdots, \partial_{x_{n-1}}, \varphi(x_1)\partial_{x_n}\}$, where*

$$\varphi(x_1) = \begin{cases} e^{-\frac{1}{|x_1|}}, & x_1 \neq 0, \\ 0, & x_1 = 0, \end{cases}$$

defined on an open domain Ω of $\mathbb{R}^n$ which contains the origin, then $\triangle_X$ is an infinitely degenerate elliptic operator on Ω.

We can found the motivations for infinitely degenerate operators from the complex geometry:

Let $\Omega \subset \mathbb{C}^k$ be a pseudo-convex domain or pseudo-convex CR manifold with smooth boundary. Consider following $\bar{\partial}$-Neumann equation

$$\bar{\partial}\bar{\partial}^* u + \bar{\partial}^*\bar{\partial} u = f, \tag{2.1.1}$$

where $\bar{\partial}^*$ is L^2-adjoint of $\bar{\partial}$. If for any $x_0 \in \bar{\Omega}$, in a neighborhood U of x_0 and on $U \cap \bar{\Omega}$, there exist $\varepsilon > 0$ and $C > 0$, such that

$$\|u\|_\varepsilon^2 \leq C(\|\bar{\partial} u\|_0^2 + \|\bar{\partial}^* u\|_0^2 + \|u\|_0^2). \tag{2.1.2}$$

Then if $f \in C^\infty(U \cap \bar{\Omega})$, we have $u \in C^\infty(U \cap \bar{\Omega})$.

The principal part L of $\bar{\partial}$-Neumann operator $\bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$ is a sum of square operator with real dimension $n = 2k - 1$, and satisfies the sub-elliptic estimate (2.1.2).

Indeed, we can choose real C^∞ vector fields X_j, $j = 1, 2, \cdots, 2k$, spanning the real and imaginary parts of holomorphic vector fields tangent to the boundary in such a way that

$$L = -\sum_{j=1}^{2k} X_j^* X_j,$$

where X_j^* is the formal adjoint of X_j.

The simplest example is the operator on the boundary of the unit ball in $\mathbb{C}^2$. After a change of variable to $\mathbb{R}^3$, the principal part L of $\bar{\partial}$-Neumann operator is

$$L = -\{(\partial_{x_1} + 2x_2\partial_{x_3})^2 + (\partial_{x_2} - 2x_1\partial_{x_3})^2\}. \tag{2.1.3}$$

Then L is finite type degenerate elliptic operator, and the sub-elliptic estimate (2.1.2) holds for $\varepsilon = \frac{1}{2}$.

In connection with (2.1.3), it is also worthwhile to recall the example of H. Lewy [35] of an operator ($\bar{\partial}$ as it acts on scalar functions), that is not locally solvable. In these coordinates Lewy's operator is

$$(\partial_{x_1} + 2x_2\partial_{x_3}) + i(\partial_{x_2} - 2x_1\partial_{x_3}). \tag{2.1.4}$$

In 1981, Fefferman-Phong proved that, for degenerate elliptic operators P, the sub-elliptic estimate (2.1.2) holds iff P is the operator with finite order degeneracy (e.g. for sum of square operator, the Hörmander condition is satisfied). That means, at points of infinite type, the sub-elliptic estimate (2.1.2) will be not satisfied. However, there were a lot of examples in complex geometry in which the boundary of pseudo-convex domain Ω has singular points with infinite type degeneracy.

Example 2.1.2 (Example for points of infinite type on the boundary)**.** *Suppose the boundary of Ω near the origin has the form*

$$Re(z_k) = \sum_{j=1}^{N} |h_j(z_1, \cdots, z_{k-1})|^2 e^{-1/(|z_1|^2+|z_2|^2+\cdots+|z_{k-1}|^2)}, \tag{2.1.5}$$

where h_j are holomorphic functions in $\mathbb{C}^{k-1}$ with an isolated zero at the origin.

In 1987, Y. Morimoto [39] (also see M. Christ [13] for general case in 1997) proved that, if infinitely degenerate elliptic operator satisfies the so-called logarithmic regularity estimate, then it is hypo-elliptic (the details please see the contents below). Later in 2002, by using sub-elliptic multipliers method, J. Kohn [27] gave a purely geometrical condition for the hypo-ellipticity at points of infinite type degeneracy on the boundary of pseudo-convex domain Ω (also see [28]-[31]).

2.1.2 Hypoellipticity and Some Applications

The first known hypoellipticity results for infinitely degenerate operators are due to Fedĭi[15] by the means of priori estimates, where the simplest example is $P = \partial_x^2 + k(x)\partial_y^2$ with $k(x) > 0$ for $x \neq 0$, $\sqrt{k(x)}$ is smooth and it may vanish to any order at the origin. Later, Kusuoka and Stroock [26] obtained the following remarkable result:

Theorem 2.1.1 (c.f. [26])**.** *Let $\varphi(\xi) \in C_b^\infty(\mathbb{R}^1)$ be a non negative even function which satisfies: $\varphi(\xi) = 0$ if and only if $\xi = 0$, $\varphi(\xi)$ is non-decreasing in $\xi \in [0, \infty)$. Define $X = (\partial x_1, \partial x_2, \varphi(x_1)\partial x_3)$ on $C^\infty(\mathbb{R}^3)$, then $\triangle_X$ is hypoelliptic on $\mathbb{R}^3$ if and only if*

$$\lim_{x_1 \to 0} x_1 \log|\varphi(x_1)| = 0,$$

where $C_b^\infty(\mathbb{R}^1) = \{u \in C^\infty(\mathbb{R}^1); u \text{ is bound}\}$.

Remark 2.1.1. *The main method in [26] is Malliavin calculus(also called stochastic calculus of variations) in stochastic process. Later, Morimoto [39] also gain the same results by using the theory of pseudo-differential operators in PDE. Also, we give the results about hypoellipticity of some other infinitely degenerate operators defined on* $\mathbb{R}^3$ *(cf. [23, 39, 41]).*

(I) *The operator*

$$L_1 = \partial_{x_1}^2 + \exp(-1/|x_1|^\sigma)\partial_{x_2}^2 + x_1^{2k}\partial_{x_3}^2,$$

where $\sigma > 0$, $k \in \mathbb{N}^+$, *then the operator* L_1 *is hypoelliptic if and only if* $\sigma < k+1$.

(II) *If* $\sigma_1, \sigma_2 > 0$, *then the operator*

$$L_2 = \partial_{x_1}^2 + \exp(-1/|x_1|^{\sigma_1})\partial_{x_2}^2 + \exp(-1/|x_1|^{\sigma_2})\partial_{x_3}^2$$

is hypoelliptic.

(III) *The operator*

$$L_3 = \partial_{x_1}^2 + \exp(-1/|x_1|^\sigma)\partial_{x_2}^2 - x_1^{2k}\partial_{x_3},$$

where $\sigma > 0$, $k \in \mathbb{N}^+$, *then the operator* L_3 *is hypoelliptic if and only if* $\sigma < 2k+2$.

Let $\{X_1, \cdots, X_m\}$ denote a system of real smooth vector fields defined on an open subset Ω of $\mathbb{R}^n$. For any positive integer k, let $X^{(k)}$ denote a matrix whose columns consist of $X_1, \cdots, X_m$, together with all vector fields of the form

$$[X_{i_1}, X_{i_2}]_{i_1,i_2=1}^m; \cdots; [X_{i_1}, [X_{i_2}, [X_{i_3}, \cdots, [X_{i_{m-1}}, X_{i_m}]]] \cdots]_{i_1,i_2,\cdots,i_m=1}^m,$$

arranged in a specified order. The symbol $[\cdot,\cdot]$ denotes the Lie bracket operation on vector fields. For any $x \in \Omega$ and $m \geq 1$, define $\lambda^{(m)}(x)$ to be the smallest eigenvalues of the matrices $[X^{(m)}(x)]^2$. Note that $\lambda^{(m)}(x)$ is independent of the choice of the basis in the space of vector fields and is also independent of the specific ordering of the columns referred to above.

Remark 2.1.2 (cf. [3])**.** $\lambda^{(m)}(x) > 0$ *for some* $m \geq 1$ *if and only if Hörmander condition holds for* X *at* $x \in \Omega$.

Definition 2.1.2 (Non-Hörmander points)**.** *We say that* $x \in \Omega$ *is a Hörmander point for the operator* $\triangle_X$ *if there is an integer* $m \geq 1$ *such that* $\lambda^{(m)}(x) > 0$. *The set of all Hörmander point is denoted by* H. *Note that the sets* H *is open in* Ω. *The points in the closed sets* H^c *will be called non-Hörmander points.*

Remark 2.1.3. *It follows from Fefferman and Phong's results in [16] that* $\triangle_X$ *is not subelliptic on* H^c.

Theorem 2.1.2 (c.f. [3])**.** *Suppose that the non-Hörmander set* H^c *of* X *is contained in a* C^2 *submanifold* M *of* Ω *satisfying* codimension $\mathrm{M} = 1$ *and* M *is non-characteristic with respect to* X. *Assume further for every* $x \in H^c$, *there exist an integer* $m \geq 1$, *an open neighborhood* U *of* x, *and an exponent* $p \in (-1, 0)$ *such that*

$$\lambda^{(m)}(y) \geq \exp\{-[d(y,M)]^p\}, \text{ for all } y \in U,$$

where $d(y,M)$ *is the Euclidean distance of* y *from* M. *Then* $\triangle_X$ *is hypoelliptic on* Ω.

Remark 2.1.4. *To prove Theorem 2.1.2, the authors used the probabilistic methods. Also, Oleinik and Radkevic [45] have shown that if the non-Hörmander set H^c of $\triangle_X$ is compact and non-characteristic, then $\triangle_X$ is hypoelliptic on Ω. However if the compactness assumption on H^c is dropped, then a further hypothesis such as the exponential degeneracy condition in Theorem 2.1.2 is required, which controls the rate at which the Hörmander condition fails as one approaches H^c.*

For a system of vector fields $X = \{X_1, \cdots, X_m\}$ defined on an open domain Ω of $\mathbb{R}^n$ with $X_j \in Op(S^1_{1,0}(\Omega))$, the PsDO of order 1, which satisfies the following global inequality

$$\int_{\mathbb{R}^n} \omega^2(\xi)|\hat{u}(\xi)|^2 d\xi \leq C\big(\|Xu\|^2_{L^2(\mathbb{R}^n)} + \|u\|^2_{L^2(\mathbb{R}^n)}\big), \ \forall\ u \in C^1_0(\Omega), \tag{2.1.6}$$

where $\hat{u}(\xi)$ is the Fourier transform of $u(x)$, C is a positive constant, and ω is a strictly positive, continuous function satisfying $\omega(\xi) \to \infty,$ as $|\xi| \to \infty$. Thus we have

Theorem 2.1.3 (c.f. [13])**.** *For a system of vector fields X, suppose that there exists a function ω satisfying*

$$\frac{\omega(\xi)}{\log(e + |\xi|^2)^{1/2}} \to \infty, \ \text{as } |\xi| \to \infty, \tag{2.1.7}$$

for which (2.1.6) *holds. Then $\triangle_X$ is hypoelliptic in Ω.*

Remark 2.1.5. (1) *The hypothesis* (2.1.7) *is the optimal condition of this type (see Theorem 2.1.1).*

One example for the operator $\partial^2_{x_1} + \partial^2_{x_2} + e^{-(2/x_1)}\partial^2_{x_3}$, in $\mathbb{R}^3$ satisfies the inequality (2.1.6) *with $w(\xi) = \log(e + |\xi|^2)^{1/2}$, and fails to be hypo-elliptic (also see the result in Theorem 2.1.1 above).*

(2) *An equivalent formulation of* (2.1.7) *is that for each $\delta > 0$ there should exist a positive constant C_δ such that for each real valued function $u \in C^2_0(\Omega)$,*

$$\int_{\mathbb{R}^n} \big(\log < \xi >\big)^2 |\hat{u}(\xi)|^2 d\xi \leq \delta\|Xu\|^2_{L^2(\mathbb{R}^n)} + C_\delta\|u\|^2_{L^2(\mathbb{R}^n)}, \tag{2.1.8}$$

where (also thereinafter) $< \xi >= (e + |\xi|^2)^{1/2}$.

(3) *If X satisfies the hypothesis in Theorem 2.1.2, then for every relatively compact open subset $U \subset\subset \Omega$ and each small $\delta > 0$ there exists $C_\delta > 0$ such that for all $u \in C^\infty_0(U)$,* (2.1.8) *holds, which leads to the hypoelliptic of $\triangle_X$ by Theorem 2.1.3.*

Now we give the sketch of the proof for Theorem 2.1.3.

Definition 2.1.3 (Symbol class $S^m_{\rho,\delta}(\Omega)$)**.** *Suppose Ω is an open set in $\mathbb{R}^n$, m is a real number and $0 \leq \rho, \delta \leq 1$. The symbol class of order m on Ω, denoted by $S^m_{\rho,\delta}(\Omega)$, is the space of functions $p \in C^\infty(\Omega \times \mathbb{R}^n)$ such that for all multi-indices α and β and every compact set $K \subset \Omega$, there is a constant $C_{\alpha,\beta,K}$ such that*

$$\sup_{x \in K} |D^\beta_x D^\alpha_\xi p(x, \xi)| \leq C_{\alpha,\beta,K}(1 + |\xi|)^{m - \rho|\alpha| + \delta|\beta|}.$$

Definition 2.1.4 (Symbol class $S^{m+}_{1,0}$)**.** *Denote by $S^{m+}_{1,0}$ the intersection, over all $\varepsilon > 0$, of all classes $S^{m+\varepsilon}_{1-\varepsilon,\varepsilon}(\mathbb{R}^n)$.*

Definition 2.1.5 (Symbol class $S^{m,k}$). *$a(x,\xi)$ belongs to the classes $S^{m,k}$ if $a \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ and satisfies*

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \le C_{\alpha,\beta} < \xi >^{m-|\beta|} (\log < \xi >)^{k+|\beta|+|\alpha|},$$

for all α, β, and $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n$, where $< \xi >= (e + |\xi|^2)^{1/2}$.

Remark 2.1.6. *It is obvious by definitions that $S^{m,k} \subset S^{m+}_{1,0}$.*

Recall that if a, b are symbols in some classes $S^m_{\rho,\delta}$ and $S^n_{\rho,\delta}$, and $\rho > \delta$, then $Op(a) \circ Op(b)$ has a symbol $a \odot b$ with an asymptotic expansion

$$a \odot b(x,\xi) \sim \sum_\alpha c_\alpha \partial_\xi^\alpha a(x,\xi) \partial_x^\alpha b(x,\xi),$$

where $c_\alpha = (\alpha!)^{-1}(-i)^\alpha$. The notation $\sim$ indicates convergence in the usual asymptotic sense: for any positive integer N, the difference between $Op(a) \circ Op(b)$ and an operator associated to the symbol $\sum_{|\alpha|<N} c_\alpha \partial_\xi^\alpha a(x,\xi) \partial_x^\alpha b(x,\xi)$ is smoothing of order $m+n-N(\rho-\delta)$ in the scale of Sobolev spaces.

Proof of Theorem 2.1.3: The detail proof of Theorem 2.1.3 can be found in Christ [13], in which we need to use the technique of micro-local analysis. Here we only give a sketch of the proof.

We divide the proof into four steps as follows.

Step 1: Let $L = -\triangle_X$. Then there exists a pseudo-differential operator G of the form

$$G = \sum_j B_j \circ X_j + \sum_j X_j^* \circ \tilde{B}_j + B_0, \tag{2.1.9}$$

where $B_0 \in Op(S^{0,2})$ and $B_j, \tilde{B}_j \in Op(S^{0,1})$ for each $j \ge 1$, such that

$$(L + G)\eta_1 \Lambda \eta_2 = \eta_1 \Lambda L \eta_2 + R, \tag{2.1.10}$$

for some R belonging to $Op(S^{-M+}_{1,0})$ for every $M < \infty$. Here Λ is a PsDO with nonconstant order whose symbol λ depends on parameters s and N_0,

$$\lambda(x,\xi) = |\xi|^s e^{-N_0 \log|\xi| \varphi(x,\xi)}, \text{ for } |\xi| \ge e, \tag{2.1.11}$$

where the function $\varphi(x,\xi) \in C^\infty\Big(\mathbb{R}^n \times (\mathbb{R}^n \backslash \{0\})\Big)$ is nonnegative and homogeneous of degree zero with respect to ξ and has compact support with respect to x. Then the non-negativity of φ implies that $\lambda \in S^{s,0} \subset S^{s+}_{1,0}$. On the other hand, η_1, η_2 are the cut-off functions in Ω, satisfying $\eta_1 \equiv 1$ in a neighborhood of supp η_2.

Step 2: Let G be a pseudo-differential operator of the form (2.1.9). Then for any fixed relatively compact subset $U \subset \Omega$, any $\delta > 0$ and any $f \in C^2$ with support in U,

$$|\langle Gf, f\rangle| \le C_\delta \int \log^2 < \xi > |\hat{f}(\xi)|^2 d\xi + \delta \sum_j \|X_j f\|^2_{L^2(\Omega)}. \tag{2.1.12}$$

Step 3: Let $L = -\triangle_X$ satisfy (2.1.6) and (2.1.7). Let s, $M \in \mathbb{R}$ be fixed. If N_0 in (2.1.11) is chosen to be sufficiently large in the definition of Λ, then from (2.1.10) and (2.1.12), we choose the cut-off function $\eta_2 \equiv 1$ on the support of u. Then for any fixed relatively compact subset $U \subset\subset \Omega$ and $u \in C_0^{s+3}(U)$, since $(L+G)\eta_1\Lambda\eta_2 u = \eta_1\Lambda L\eta_2 u + Ru = \eta_1\Lambda Lu + Ru$, and then if we choose $v = \eta_1\Lambda u \in C_0^2$ we can prove that

$$\langle (L+G)v, v\rangle = \sum_j \|X_j v\|^2 + \|v\|^2 + O\Big(\|v\| \cdot \|Gv\|\Big).$$

Thus from (2.1.12),

$$\begin{aligned}\sum_j \|X_j v\|^2 &\le \|\eta_1\Lambda Lu\|^2 + \|Ru\|^2 + C(\|v\|^2 + \|Gv\|^2)\\ &\le \|\eta_1\Lambda Lu\|^2 + \|Ru\|^2 + C\|v\|^2 + C_\delta \int \log^2 <\xi> |\hat{v}(\xi)|^2 d\xi + \delta \sum_j \|X_j v\|^2.\end{aligned}$$

Since $\|v\|^2$ can be majorized by $\int \log^2 <\xi> |\hat{v}(\xi)|^2 d\xi$. Thus we choose $\delta < 1$ to get

$$\sum_j \|X_j v\|^2 \le C_1 \int \log^2 <\xi> |\hat{v}(\xi)|^2 d\xi + \|\eta_1\Lambda Lu\|^2 + C_2\|u\|^2_{H^{-M}}.$$

Then from the condition (2.1.7) in Theorem 2.1.3, we have

$$\sum_j \|X_j v\|^2 \ge A \int \log^2 <\xi> |\hat{v}(\xi)|^2 d\xi - C_A\|v\|^2,$$

for arbitrarily large A. That implies

$$\int \log^2 <\xi> |\hat{v}(\xi)|^2 d\xi \le \bar{C}\|\eta_1\Lambda Lu\|^2 + \bar{C}\|u\|^2_{H^{-M}} + \bar{C}\|v\|^2,$$

for some constant $\bar{C} > 0$. Finally we can deduce that

$$\|\eta_1\Lambda u\|_{L^2} \le C_3\|\eta_1\Lambda Lu\|_{L^2} + C_4\|u\|_{H^{-M}}, \text{ for any } u \in C_0^{s+3}(U). \tag{2.1.13}$$

Step 4: Using the estimate (2.1.13), similar to the finial part of the proof of Theorem 1.1.1, we can complete the proof of Theorem 2.1.3. □

2.1.3 Logarithmic Regularity Estimate

Definition 2.1.6 (Logarithmic regularity estimate). *Let $\Omega \subset \mathbb{R}^n$ an open domain, and $X = (X_1, X_2, \cdots, X_m)$ be an infinitely degenerate system of vector fields on Ω. If for $s > 0$, there exists $C > 0$ such that*

$$\|(\log\Lambda)^s u\|^2_{L^2(\Omega)} \le C\big(\|Xu\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\Omega)}\big), \text{ for any } u \in C_0^\infty(\Omega), \tag{2.1.14}$$

where $\Lambda = (e^2 + |\nabla|^2)^{1/2}$. Then we say that X satisfies logarithmic regularity estimate.

Remark 2.1.7. *From Theorem 2.1.3, if X satisfies the logarithmic regularity estimate* (2.1.14) *with $s > 1$, Then $\triangle_X$ is hypo-elliptic. Also, we have a very simple example which satisfies the estimate* (2.1.8) *but not satisfies* (2.1.14) *for any $s > 1$. It is the system of vector fields in $\mathbb{R}^3$ such as $X_1 = \partial_{x_1}, X_2 = \partial_{x_2}, X_3 = \exp\big(-(|x_1||\log|x_1||)^{-1}\big)\partial_{x_3}$ (cf. [40]).*

Let X_J denote the repeated commutator

$$[X_{j_1}, [X_{j_2}, [X_{j_3}, \cdots [X_{j_{k-1}}, X_{j_k}] \cdots]]],$$

for $J = (j_1, \cdots, j_k)$, $j_i \in \{1, \cdots, m\}$, and $|J| = k$. For $k \geq 1$, we take

$$G(x,k) = \min_{\xi \in \mathbb{S}^{n-1}} \sum_{|J| \leq k} |X_J(x,\xi)|^2, \qquad g(t,j,k,x_0) = G((\exp tX_j)(x_0), k),$$

where $(\exp tX_j)(x_0)$ denotes the integral curve of X_j starting from $x_0 \in \Gamma$. Here $\Gamma = \{x \in \Omega; \exists\ \xi \in \mathbb{S}^{n-1}, X_J(x,\xi) = 0, \text{ for any } J\}$, and $g_I^{j,k}(x_0) = \frac{1}{|I|}\int_I g(t,j,k,x_0)dt$ is the mean value of $g(t,j,k,x_0)$ on the interval I.

Theorem 2.1.4 (Sufficient condition)**.** *If $s > 0$ and there exists $\varepsilon_1 > 0$ such that*

$$\inf_{\delta>0, k\in\mathbb{N}, \mu>0, 1\leq j\leq m} \Big\{ \sup \big(|I|^{\frac{1}{s}} |\log g_I^{j,k}(x_0)|\big); I \subset (-\mu,\mu), g_I^{j,k}(x_0) < \delta \Big\} < \varepsilon_1, \tag{2.1.15}$$

for any $x_0 \in \Gamma$. Then there exist constants $C_0 > 0$ which is independent with ε_1 and C_{ε_1} such that

$$\|(\log \Lambda)^s u\|^2_{L^2(\Omega)} \leq C_0 \varepsilon_1^{2s} \int_\Omega |Xu|^2 dx + C_{\varepsilon_1} \|u\|^2_{L^2(\Omega)}, \text{ for any } u \in C_0^\infty(\Omega). \tag{2.1.16}$$

Lemma 2.1.1 (Sawyer's lemma, c.f. [53])**.** *Let I_0 be an open interval in $\mathbb{R}^1_x$ and let $V(t), W(t) \geq 0$ belong to $L^1_{loc}(I_0)$. Then we have the estimate*

$$\int_{I_0} V(t)|v(t)|^2 dt \leq C \int_{I_0} \big(W(t)|v(t)|^2 + |v'(t)|^2\big) dt,$$

for all $v \in C_0^1(I_0)$ with a constant $C > 0$ if and only if

$$V_I \leq A(3W_{3I} + 2|I|^{-2}), \text{ for any interval } I \text{ with } 3I \subset I_0,$$

holds with a constant $A > 0$. Here $3I$ denotes the interval with the same center as I but with length three times, $U_{I_1} = \frac{1}{|I_1|}\int_{I_1} U(x)dx$ denotes the mean value of function $U(x)$ on the interval I_1.

Brief Proof of Theorem 2.1.4: Here we only give the sketch of proof for Theorem 2.1.4, the details please see [42].

It follows from (2.1.15) that there exist some $j \in \{1, \cdots, m\}$, $\delta > 0$, $k \in \mathbb{N}$ and $\mu > 0$ such that

$$|\log g_I^{j,k}(x_0)|^{2s} \leq (2\varepsilon)^{2s} |I|^{-2}, \quad \text{if } I \subset (-\mu,\mu) \text{ and } g_I^{j,k}(x_0) < \delta.$$

Take the new local coordinates $x = (x_1, x')$ in a neighborhood of x_0 such that $x_0 = (0,0)$ and the line $x' =$ constant vector in $\mathbb{R}^{n-1}$ is the integral curve of X_j starting from $(0,x')$. Since $G(x;k)$ is continuous, we have

$$|\log g_I^{j,k}(0,x')|^{2s} \leq (4\varepsilon)^{2s}|I|^{-2}, \quad \text{if } I \subset (-\mu,\mu)\ |x'| < \mu, \text{ and } g_I^{j,k}(0,x') < \delta,$$

by taking other small $\mu, \delta > 0$ if necessary. For a moment we consider x' as parameters. Let λ be a large parameter satisfying $\lambda > 1/\delta$. If $g_I^{j,k}(0,x')\lambda < 1$, then we have $-\log g_I^{j,k}(0,x') > \log \lambda$ and hence

$$(\log \lambda)^{2s} \leq (4\varepsilon)^{2s}(|I|^{-2} + g_I^{j,k}(0,x')\lambda^2), \ \forall I \subset (-\mu,\mu). \tag{2.1.17}$$

When $g_I^{j,k}(0,x')\lambda \geq 1$, this is also true for $\lambda \geq \lambda_0$ if λ_0 is chosen sufficiently large, depending on ε.

Let $V(t) = (\log \lambda)^{2s}$ and $W(t) = g(t; j, (0, x'))\lambda^2 = G(t, x'; k)\lambda^2$ in Lemma 2.1.1 and replace $3I$ by I, we see that (2.1.17) implies

$$\int (\log \lambda)^{2s} |v(t)|^2 dt \leq C_0 \varepsilon^{2s} \int (|D_t v(t)|^2 + G(t, x'; k)\lambda^2 |v(t)|^2) dt, \ \forall v(t) \in C_0^1\big((-\mu, \mu)\big), \tag{2.1.18}$$

where $C_0 > 0$ is a constant independent of ε.

Also, it is well known that

$$\sum_{|J| \leq k} \|\Lambda^{\delta - 1} X_J u\|^2 \leq C\{(\Delta_X u, u) + \|u\|^2\},$$

for some $0 < \delta = \delta(k) \leq 1/2$. If we set

$$q(x_1, x', \xi')\big(\sum_{|J| \leq k} \xi^{2\delta - 2} |X_J u|^2 \big)\big|_{\xi_1 = 0},$$

in our local coordinates near x_0, then we have $q(x_1, x', \xi') - G(x; k) \geq 0$ on $\xi' \in \mathbb{S}^{n-2}$ and

$$\|D_t u\|^2 + (q^w(t, x', D')u, u) \leq C\{(\Delta_X u, u) + \|u\|^2\},$$

where q^w denotes the pseudo-differential operator with Weyl symbol in $\mathbb{R}^{n-1}_{x'}$.

If $\tilde{q}(x_1, x', \xi') = q(x_1, x', \xi')|\xi'|^{-2\delta}$, then in view of the Littlewood-Paley decomposition in $\mathbb{R}^{d-1}_{\xi'}$ we may replace the second term by $(\tilde{q}^w(t, x', D')\lambda^2 u, u)$, provided that the support of the partial Fourier transform of $u(x, t')$ with respect to x' is contained in $\{\lambda^{1/\delta} \leq |\xi|' \leq 2\lambda^{1/\delta}\}$. Though G is not smooth enough in general, the Wick approximation of $\tilde{q}^w$ gives

$$(\tilde{q}^w(t, x', D')\lambda^2 u, u) \geq \big(G(t, x'; k)\lambda^2 u, u\big) - C\|u\|^2.$$

Hence (2.1.18) leads us to (2.1.16) for u with supp u contained in a small neighborhood of x_0. Finally, the usual covering argument shows (2.1.16) for the general u. $\square$

Example 2.1.3. *Let $s > 0$, and*

$$\varphi(x_1) = \begin{cases} e^{-\frac{1}{|x_1|^{1/s}}}, & x_1 \neq 0, \\ 0, & x_1 = 0. \end{cases}$$

Then $X = (\partial_{x_1}, \cdots, \partial_{x_{n-1}}, \varphi(x_1)\partial_{x_n})$ is infinitely degenerate on the surface $\Gamma = \{x_1 = 0\}$ and X satisfies the logarithmic regularity estimate (2.1.14).

Proof of Example 2.1.3: First, from the fact $\varphi^{(n)}(x_1)|_{x_1=0} = 0$, for all $n \in N^+$, we can obtain that X is infinitely degenerate on the surface $\Gamma = \{x_1 = 0\}$. Next, let

$$A = \inf_{\delta > 0, k \in \mathbb{N}, \mu > 0, 1 \leq j \leq m} \Big\{ \sup \big(|I|^{\frac{1}{s}} |\log g_I^{j,k}(x_0)|\big); I \subset (-\mu, \mu), g_I^{j,k}(x_0) < \delta \Big\}, \tag{2.1.19}$$

then, we know

$$A \leq \inf_{\mu > 0} \Big\{ \sup \big(|I|^{\frac{1}{s}} |\log g_I^{1,1}(x_0)|\big); I \subset (-\mu, \mu), g_I^{j,k}(x_0) < 1 \Big\}. \tag{2.1.20}$$

Now, we calculate $g_I^{1,1}(x_0)$.

$$G(x,1) = \min_{\xi\in\mathbb{S}^{n-1}} \sum_{|J|\le 1} |X_J(x,\xi)|^2 = \min_{\xi\in\mathbb{S}^{n-1}} \big(\sum_{j=1}^{n-1} \xi_j^2 + \varphi^2(x_1)\xi_n^2\big),$$

where $x=(x_1,\cdots,x_n)$, $\xi=(\xi_1,\cdots,\xi_n)$, $(x,\xi)=\sum_{j=1}^n x_j\xi_j$.

Thus for any $x_0\in\Gamma=\{x_1=0\}$, suppose $x_0=(0,x_2,\cdots,x_n)$, then we gain

$$\exp(tX_1)(x_0) = (t,x_2,\cdots,x_n).$$

Since $\varphi(x_1)\le 1$, by direct calculation, we have

$$g(t,1,1,x_0) = G((\exp tX_1)(x_0),1) = \varphi^2(t).$$

Then

$$g_I^{1,1}(x_0) = \frac{1}{|I|}\int_I g(t,1,1,x_0)dt = \frac{1}{|I|}\int_I e^{-\frac{2}{|t|^{1/s}}}dt < 1.$$

So, (2.1.20) can be written as

$$A \le \inf_{\mu>0} \sup_{I\subset(-\mu,\mu)} \big(|I|^{\frac{1}{s}}|\log g_I^{1,1}(x_0)|\big). \tag{2.1.21}$$

Then, we estimate

$$\sup_{I\subset(-\mu,\mu)} \big(|I|^{\frac{1}{s}}|\log g_I^{1,1}(x_0)|\big).$$

For any interval $I=(a,b)\subset(-\mu,\mu)$, we need consider following three cases:

(i) $ab=0$. By the symmetry of $g_I^{1,1}(x_0)$, we suppose that $0=a<b<\mu$, then

$$|I|^{\frac{1}{s}}|\log g_I^{1,1}(x_0)| = -b^{\frac{1}{s}}\log\Big(\frac{1}{b}\int_0^b e^{-\frac{2}{|t|^{1/s}}}dt\Big) \le -b^{\frac{1}{s}}\log\Big(\frac{1}{2}e^{-\frac{2}{|\frac{b}{2}|^{1/s}}}\Big) \le 2^{\frac{1}{s}+1} + \mu^{\frac{1}{s}}\log 2.$$

(ii) $ab>0$. By the symmetry of $g_I^{1,1}(x_0)$, we suppose that $0<a<b<\mu$, then

$$\begin{aligned}
&|I|^{\frac{1}{s}}|\log g_I^{1,1}(x_0)| \\
&\quad = -(b-a)^{\frac{1}{s}}\log\Big(\frac{1}{b-a}\int_a^b e^{-\frac{2}{|t|^{1/s}}}dt\Big) \le -(b-a)^{\frac{1}{s}}\log\Big(\frac{1}{b}\int_0^b e^{-\frac{2}{|t|^{1/s}}}dt\Big) \\
&\quad \le -(b-a)^{\frac{1}{s}}\log\Big(\frac{1}{2}e^{-\frac{2}{|\frac{b}{2}|^{1/s}}}\Big) \le (b-a)^{\frac{1}{s}}b^{-\frac{1}{s}}2^{\frac{1}{s}+1} + (b-a)^{\frac{1}{s}}\log 2 \\
&\quad \le 2^{\frac{1}{s}+1} + \mu^{\frac{1}{s}}\log 2.
\end{aligned}$$

(iii) $ab<0$. By the symmetry of $g_I^{1,1}(x_0)$, we suppose that $0<-a<b<\mu$, then

$$\begin{aligned}
&|I|^{\frac{1}{s}}|\log g_I^{1,1}(x_0)| \\
&\quad = -(b-a)^{\frac{1}{s}}\log\Big(\frac{1}{b-a}\int_a^b e^{-\frac{2}{|t|^{1/s}}}dt\Big) \le -(b-a)^{\frac{1}{s}}\log\Big(\frac{1}{2b}\int_0^b e^{-\frac{2}{|t|^{1/s}}}dt\Big) \\
&\quad \le -(b-a)^{\frac{1}{s}}\log\Big(\frac{1}{4}e^{-\frac{2}{|\frac{b}{2}|^{1/s}}}\Big) \le (b-a)^{\frac{1}{s}}b^{-\frac{1}{s}}2^{\frac{1}{s}+1} + 2(b-a)^{\frac{1}{s}}\log 2 \\
&\quad \le 2^{\frac{2}{s}+1} + \mu^{\frac{1}{s}}2^{\frac{1}{s}+1}\log 2.
\end{aligned}$$

From above discussion, we know

$$\sup_{I\subset(-\mu,\mu)}\left(|I|^{\frac{1}{s}}|\log g_I^{1,1}(x_0)|\right)\leq 2^{\frac{2}{s}+1}+\mu^{\frac{1}{s}}2^{\frac{1}{s}+1}\log 2.$$

Taking $\varepsilon_1=2^{\frac{2}{s}+1}$, we have

$$\inf_{\mu>0}\sup_{I\subset(-\mu,\mu)}\left(|I|^{\frac{1}{s}}|\log g_I^{1,1}(x_0)|\right)\leq\varepsilon_1. \tag{2.1.22}$$

Then, (2.1.19), (2.1.21) and (2.1.22) imply

$$\inf_{\delta>0,k\in\mathbb{N},\mu>0,1\leq j\leq m}\left\{\sup\left(|I|^{\frac{1}{s}}|\log g_I^{j,k}(x_0)|\right);I\subset(-\mu,\mu),g_I^{j,k}(x_0)<\delta\right\}<\varepsilon_1.$$

By Theorem 2.1.4, we can prove that X satisfies the logarithmic regularity estimate (2.1.14). $\square$

Example 2.1.4. *The system of vector fields $X=(\partial_{x_1},\cdots,\partial_{x_{n-1}},\varphi(x_1)\partial_{x_n})$, for $n\geq 2$, where $s>0$ and*

$$\varphi(x_1)=\begin{cases}e^{-\frac{1}{|x_1\sin(\frac{\pi}{x_1})|^{1/s}}}, & x_1\neq 0,\\ 0, & x_1=0.\end{cases}$$

Then X is infinitely degenerate on $\Gamma=\bigcup_{j\in\mathbb{Z}_+}\Gamma_j$, for $\Gamma_j=\{x_1=\frac{1}{j}\}$, $j\geq 1$, and $\Gamma_0=\{x_1=0\}$.

Example 2.1.5. *The system of vector fields $X=(\partial_{x_1},\cdots,\partial_{x_{n-1}},\varphi(x_1,x_2)\partial_{x_n})$, for $n\geq 3$, where $k\geq 1$, $s>0$ and*

$$\varphi(x_1,x_2)=\begin{cases}e^{-\frac{1}{|x_1|^{1/s}}}{x_2}^k, & x_1\neq 0,\\ 0, & x_1=0.\end{cases}$$

Then X is infinitely degenerate on the surface $\{x_1=0\}$.

Proposition 2.1.1 (Controllability, c.f.[43])**.** *Let Ω be a bounded and connected open subdomain of $\mathbb{R}^n$, $\mathfrak{X}(X_1,\cdots,X_m)$ be the Lie algebra spanned by the system of vector fields X and their commutators. If $\triangle_X+c(x)$ is hypoelliptic in Ω for any $c\in C^\infty(\Omega)$, then any two points of Ω can be linked by continuous curve made of a finite numbers of the integral paths of vector fields belonging to $\mathfrak{X}(X_1,\cdots,X_m)$.*

Remark 2.1.8. (1) *It should be noted that the controllability can be deduced from the hypoellipticity of $\triangle_X+c(x)$. Conversely, the controllability does not imply the hypoellipticity of $\triangle_X$. The first Example 2.1.3 with $0<s\leq 1$ satisfies the logarithmic regularity estimate* (2.1.14)*, which satisfies the controllability but not the hypoellipticity.*

(2) *The result of controllability will enable us to define the metric (C-C metric) associated with X. This metric might set light aglow in the analysis for infinitely degenerate vector fields X.*

Now, we give an example to show that the C-C metric induced by the infinitely degenerate operator $\triangle_X$ may be not doubling.

Lemma 2.1.2 (c.f. [50]). *Let $X = (\partial_x, \varphi(x)\partial_y)$ be a vector fields in $\mathbb{R}^2$. Here the function $\varphi(x) \in C^\infty(\mathbb{R})$ is even, $\varphi(x) > 0$ if $x \neq 0$, and $\varphi(x)$ can vanish at $x = 0$ together with all its derivatives. If $\triangle_X$ is a hypoelliptic operator, define the box*

$$Q_r(x, y) = [x - r, x + r] \times [y - r\varphi(r/2), y + r\varphi(r/2)],$$

and

$$\tilde{Q}_r(x, y) = [x - r/2, x + 3r/4] \times [y - r\varphi(r/2)/4, y + r\varphi(r/2)/4].$$

Then for any $r > 0$, the balls $B\big((0, y), r\big) = \{z \in \mathbb{R}^2, d_1\big(z; (0, y)\big) < r\}$ satisfy

$$\tilde{Q}_r(0, y) \subset B\big((0, y), r\big) \subset Q_r(0, y), \text{ and then } r^2\varphi(r/2)/8 \leq |B\big((0, y), r\big)| \leq 4r^2\varphi(r/2),$$

where d_1 is the C-C metric definded by Definition 1.2.2.

Proposition 2.1.2. *Let $X = (\partial_x, \varphi(x)\partial_y)$ in $\mathbb{R}^2$, where $s > 0$ and*

$$\varphi(x) = \begin{cases} e^{-\frac{1}{|x|^{1/s}}}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Then $(\mathbb{R}^2, d_1)$ is non-doubling, where d_1 is the C-C distance induced by the vector fields X.

Proof: From Lemma 2.1.2, we have

$$\frac{B\big((0, y), 2r\big)}{B\big((0, y), r\big)} \geq \frac{\varphi(r)}{8\varphi(r/2)} = \frac{1}{8} e^{(2^{\frac{1}{s}} - 1)/r^{1/s}}.$$

This means

$$\lim_{r \to 0+} \frac{B\big((0, y), 2r\big)}{B\big((0, y), r\big)} = +\infty.$$

That means $(\mathbb{R}^2, d_1)$ is non-doubling. $\square$

Proof of Lemma 2.1.2: Since the operator matrix only depends on the first variable, it is enough to prove the statement for $y = 0$. To prove the inclusion in, first note, that any horizontal line segment is a admissible curve. Next, consider a point $p = (x_0, \frac{r}{4}\varphi(r/2))$ on the "top side" of $\tilde{Q}(0, 0)$, where $r/2 \leq x_0 \leq 3r/4$. Let a admissible curve $\gamma = \gamma_1 \cup \gamma_2$ connect the origin to the point p. Here, γ_1 is a horizontal segment, connecting $(0, 0)$ to $(x_0, 0)$ and $\gamma_2(t)$ is defined as follows

$$\gamma_2(t) = \big(x_0, \varphi(r/2)t\big), \ t \in [0, \frac{r}{4}].$$

Since φ is an increasing function on $\mathbb{R}_+$, therefore,

$$d_1\big((0, 0), p\big) \leq |x_0 - \frac{r}{2}| + \frac{r}{4} < r.$$

Therefore, $p \in B(0, r)$. Moreover, it is clear that any other point in $\tilde{Q}_r$ can be connected to the origin by a similarly constructed curve, so that the distance to the origin is less than r. This concludes the proof that $\tilde{Q}_r \subset B(0, r)$. To show the other inclusion, let $\gamma(t)$ be the minimizing curve connecting the origin to any point on the boundary ∂Q_r. First, let the point (x, y) belong to the top or the bottom edge of ∂Q_r, i.e. $|y| = r\varphi(r/2)$. Without loss of generality we can also assume, $x \geq 0$. The curve $\gamma(t)$ is thus an admissible curve satisfying

$$\gamma(0) = (0, 0), \ \gamma(T) = (x, y), \ T = d_1\big((0, 0), (x, y)\big).$$

Then we have

$$r\varphi(r/2) = |y-0| = \Big|\int_0^T \gamma_2'(t)dt\Big| \le \int_0^T |\gamma_2'(t)|dt \le \int_0^T \varphi\big(\gamma_1(t)\big)dt. \tag{2.1.23}$$

In order to estimate $|\gamma_1(t)|$, we first note the following

$$T = d_1\big((0,0),(0,y)\big) = d_1\big((0,0),(\gamma_1(t),\gamma_2(t))\big) + d_1\big((\gamma_1(t),\gamma_2(t)),(0,y)\big).$$

Moreover, we have $|X-Y| \le d_1(X,Y)$, $X, Y \in \mathbb{R}^2$. Thus, we obtain

$$T = d_1\big((0,0),(0,y)\big) \ge \sqrt{\gamma_1^2(t)+\gamma_2^2(t)} + \sqrt{\gamma_1^2(t)+(y-\gamma_2(t))^2} \ge 2|\gamma_1(t)|,$$

or $|\gamma_1(t)| \le T/2$ and therefore from (2.1.23), $r\varphi(r/2) \le T\varphi(T/2)$. By assumption, the function $x\varphi(x)$ is strictly increasing for $x > 0$ and thus $T \ge r$. Now, if the point $(x,y) \in \partial Q_r$ satisfies $|x| = r$, it is obvious that $d_1\big((0,0),(0,y)\big) \ge r$. This completes the proof. $\square$

2.2 Boundary-Value Problems

2.2.1 Logarithmic Sobolev Inequality

Theorem 2.2.1 (Logarithmic Sobolev inequality, c.f [42]). *Suppose that the system of vector fields $X = (X_1, \cdots, X_m)$ satisfies the logarithmic regularity estimate (2.1.14) for $s > \frac{1}{2}$. Then there exists $C_0 > 0$ such that*

$$\int_\Omega |u|^2 |\log(\frac{|u|}{\|u\|_{L^2(\Omega)}})|^{2s-1} dx \le C_0\Big[\int_\Omega |Xu|^2 dx + \|u\|^2_{L^2(\Omega)}\Big], \text{ for all } u \in H^1_{X,0}(\Omega). \tag{2.2.1}$$

The proof of Theorem 2.2.1 depends on the following lemma:

Lemma 2.2.1. *Let $\sigma_2 > 0, B > 0$, $\{v_j\}_{j\in\mathbb{N}}$ be the sequence of $H^1_{X,0}(\Omega)$ satisfying*

$$\int_\Omega |v_j|^2 |\log|v_j||^{\sigma_2} \le B.$$

Then for $\sigma_1 \in [0, \sigma_2)$, $\{|v_j|^2|\log|v_j||^{\sigma_1}\}$ is uniformly integrable and there exists a convergent sub-sequence v_{j_k} such that there exists $v_0 \in H^1_{X,0}(\Omega)$, and

$$\lim_{k\to\infty} \int_\Omega |v_{j_k}|^2 |\log|v_{j_k}||^{\sigma_1} dx = \int_\Omega |v_0|^2 |\log|v_0||^{\sigma_1} dx.$$

Proof: We prove that, for any $\varepsilon > 0$, there exists $\delta > 0$ such that if $E \subset \Omega, \mu(E) < \delta$,

$$\int_E |v_j|^2 |\log|v_j||^{\sigma_1} < \varepsilon, \ \forall j.$$

But for any $\varepsilon > 0$, there exists $t_0 > e^2$ such that

$$\frac{1}{\log^{\sigma_2-\sigma_1} t} < \varepsilon, \text{ for all } t \ge t_0.$$

Take now $\delta = \varepsilon(t_0^2 \log^{\sigma_1} t_0)^{-1}$, $\mu(E) < \delta$, and

$$A_j = E \cap \{|v_j| \le t_0\}, \ B_j = E \cap \{|v_j| > t_0\},$$

then

$$\int_{A_j} |v_j|^2 |\log |v_j||^{\sigma_1} \leq t_0^2 \log^{\sigma_1} t_0 \mu(A_j) < \varepsilon,$$

and

$$\int_{B_j} |v_j|^2 |\log |v_j||^{\sigma_1} \leq \varepsilon \int_{B_j} |v_j|^2 |\log |v_j||^{\sigma_2} < \varepsilon B.$$

Then $\{|v_j|^2 |\log |v_j||^{\sigma_1}\}$ is uniformly integrable and there exists a convergent sub-sequence v_{j_k} such that

$$\lim_{k\to\infty} \int_\Omega |v_{j_k}|^2 |\log |v_{j_k}||^{\sigma_1} dx = \int_\Omega |v_0|^2 |\log |v_0||^{\sigma_1} dx.$$

□

Let (Ω, Σ, μ) be a measure space, and f be a measurable function with real or complex values on Ω. The distribution function of f is defined for $t > 0$ by

$$\lambda_f(t) = \mu \{x \in \Omega : |f(x)| > t\}.$$

Then we have

(I). λ_f is decreasing and right continuous;
(II). If $f \leq g$, then $\lambda_f \leq \lambda_g$;
(III). If $|f_n|$ increases to $|f|$, then λ_{f_n} increases to λ_f;
(IV). If $f = g + h$, then $\lambda_f(t) \leq \lambda_g(\frac{1}{2}t) + \lambda_h(\frac{1}{2}t)$.

In fact, λ_f defines a negative Borel measure ν on $(0,\infty)$ such that

$$\nu((a,b]) = \lambda_f(b) - \lambda_f(a) = -\mu\Big(\{x; a < |f(x)| \leq b\}\Big) = -\mu\Big(|f|^{-1}((a,b])\Big).$$

Thus we use the Lebesgue-Stieltjes integral to get the following formula (cf. Folland [17] Proposition 6.23):

If $\lambda_f(t) < \infty$ for all $t > 0$ and ϕ is a nonnegative Borel measurable function on $(0,\infty)$, then

$$\int_\Omega \phi \circ |f| d\mu = -\int_0^\infty \phi(t) d\lambda_f(t). \tag{2.2.2}$$

If $\phi \in C^1$, and $\phi(t)\lambda_f(t) \to 0$ as $t \to 0$ and $t \to \infty$ respectively, then

$$\int_\Omega \phi \circ |f| d\mu = -\int_0^\infty \phi(t) d\lambda_f(t) = \int_0^\infty \phi'(t) \lambda_f(t) dt. \tag{2.2.3}$$

Now, let us give the proof of Theorem 2.2.1.

Proof of Theorem 2.2.1: Take $v \in H^1_{X,0}(\Omega)$, we use the same notation for the 0 extension of v, i.e. $v \in H^1_X(\mathbb{R}^n)$. As in the classical case, there exists a mollifier family $\{\rho_\varepsilon, \varepsilon > 0\}$ such that

$$\rho_\varepsilon * v \in C_0^\infty, \ \lim_{\varepsilon\to 0} \rho_\varepsilon * v = v \text{ in } L^2, \ \text{ and } \|X(\rho_\varepsilon * v)\|_{L^2} \leq C\{\|Xv\|_{L^2} + \|v\|_{L^2}\}.$$

Also

$$\|(\log\Lambda)^s(\rho_\varepsilon * v)\|^2_{L^2} \leq C\{\|(\log\Lambda)^s v\|_{L^2} + \|v\|^2_{L^2}\},$$

with C independent of ε. By using (2.1.14) and Lemma 2.2.1, we need only to prove the following estimate:

$$\int_\Omega |v|^2 |\log(\frac{|v|}{\|v\|_{L^2(\Omega)}})|^{2s-1} dx \le C_0 \|(\log \Lambda)^s v\|^2_{L^2}, \ \forall v \in C_0^\infty(\Omega). \tag{2.2.4}$$

By the homogenization, we prove (2.2.4) for $v \in C_0^\infty(\Omega)$ and $\|v\|_{L^2} = 1$. Since $2s-1>0$, we have

$$\begin{aligned}\int_\Omega |v|^2 |\log |v||^{2s-1} dx &\le C|\Omega| + \int_{|v|\ge e} |v|^2 |\log|v||^{2s-1} dx \\ &\le C_0 + \int_\Omega |v|^2 \log^{2s-1} < v > dx,\end{aligned} \tag{2.2.5}$$

where $< v >= (e^2 + |v|^2)^{1/2}$.

Since Ω is bounded, $v \in L^\infty(\Omega)$ and $2s-1>0$, we have from the formulas (2.2.2) and (2.2.3) that

$$\begin{aligned}&\int_\Omega |v|^2 \log^{2s-1} < v > dx = -\int_0^\infty \lambda^2 \log^{2s-1} < \lambda > d\mu\{|v| > \lambda\} \\ &= \int_0^\infty \big(2\lambda \log^{2s-1} < \lambda > + (2s-1)\frac{\lambda^3}{< \lambda >^2} \log^{2s-2} < \lambda > \big)\mu(|v| > \lambda) d\lambda,\end{aligned}$$

where $\mu(\cdot)$ is the Lebesgue measure. Since $\frac{\lambda^3}{<\lambda>^2} \le \lambda$, $\log < \lambda > \ge 1$, we have that

$$\int_\Omega |v|^2 |\log|v||^{2s-1} dx \le C_0 + C_s \int_0^\infty \lambda \log^{2s-1} < \lambda > \mu(|v| > \lambda) d\lambda. \tag{2.2.6}$$

So we need to estimate the second term of right hand side of (2.2.5). For $A > 0$ we set $v = v_{1,A} + v_{2,A}$ with $\hat{v}_{1,A} = \hat{v}(\xi) \mathbf{1}_{\{|\xi| \le e^A\}}$. Then

$$\mu\{|v| > \lambda\} \le \mu\{|v_{1,A}| > \frac{\lambda}{2}\} + \mu\{|v_{2,A}| > \frac{\lambda}{2}\}.$$

For the first term we have

$$\|v_{1,A}\|_{L^\infty} \le \|\hat{v}_{1,A}\|_{L^1} \le \|v\|_{L^2} \|\mathbf{1}_{\{|\xi|\le e^A\}}\|_{L^2} \le C_n e^{\frac{n}{2}A}.$$

Choose now $A_\lambda = \frac{2}{n} \log\left(\frac{\lambda}{4C_n}\right)$, we have $\mu\{|v_{1,A_\lambda}| > \frac{\lambda}{2}\} = 0$, hence

$$\begin{aligned}&\int_0^\infty \lambda \log^{2s-1} < \lambda > \mu(|v| > \lambda) d\lambda \\ &\quad \le C_0 + C_s \int_e^\infty \lambda \log^{2s-1} \lambda \mu(|v| > \lambda) d\lambda \\ &\quad \le C_0 + C_s \int_e^\infty \lambda \log^{2s-1} \lambda \mu(|v_{2,A_\lambda}| > \frac{\lambda}{2}) d\lambda \\ &\quad \le C_0 + 2C_s \int_e^\infty \frac{\log^{2s-1}\lambda}{\lambda} \|v_{2,A_\lambda}\|^2_{L^2} d\lambda \\ &\quad \le C_0 + 2C_s \int_e^\infty \frac{\log^{2s-1}\lambda}{\lambda} \int_{\{\xi \in \mathbb{R}^n; |\xi| \ge e^{A_\lambda}\}} |\hat{v}(\xi)|^2 d\xi d\lambda.\end{aligned}$$

Now $|\xi| \geq e^{A_\lambda}$ implies that $\lambda \leq 4C_n < |\xi| >^{n/2}$. By using Fubini theorem we have

$$\begin{aligned}
&\int_0^\infty \lambda \log^{2s-1} < \lambda > \mu(|v| > \lambda) d\lambda \\
&\leq C_0 + 2C_s \int_{\mathbb{R}^n} |\hat{v}(\xi)|^2 \int_e^{4C_n<|\xi|>^{n/2}} \frac{\log^{2s-1}\lambda}{\lambda} d\lambda d\xi \\
&\leq C_0 + 2C_s \int_{\mathbb{R}^n} |\hat{v}(\xi)|^2 \log^{2s}(4C_n < |\xi| >^{n/2}) d\xi \\
&\leq C_s \int_{\mathbb{R}^n} |\hat{v}(\xi)|^2 \log^{2s} < |\xi| > d\xi = C_s \|(\log \Lambda)^s v\|^2_{L^2(\Omega)}.
\end{aligned}$$

Thus we have proved (2.2.4) by using (2.2.6). □

Proposition 2.2.1. *Let Ω be an open bounded domain in $\mathbb{R}^n$ and the system of vector fields X satisfy the logarithmic regularity estimate* (2.1.14) *with $s > 1$, then the embedding from $H^1_{X,0}(\Omega)$ to $L^2(\Omega)$ is compact.*

Proof: Suppose $\{u_k\}$ is a sequence in $H^1_{X,0}(\Omega)$ with $\|u_k\|_{H^1_{X,0}(\Omega)} \leq C < \infty$. From logarithmic Sobolev inequality (Theorem 2.2.1), we know $\int_\Omega |u_k|^2 \big|\log|u_k|\big|^{2s-1} dx$ is bound. Then by using the result in Lemma 2.2.1, we can obtain that there exists a convergent sub-sequence u_{j_k} in $H^1_{X,0}(\Omega)$, which means that the embedding from $H^1_{X,0}(\Omega)$ to $L^2(\Omega)$ is compact. □

Now using the result of controllability (see Proposition 2.1.1) and the embedding theorem (see Proposition 2.2.1), we have following Poincaré inequality.

Proposition 2.2.2 (Poincaré inequality)**.** *Suppose that the system of vector fields X satisfies the logarithmic regularity estimate* (2.1.14) *with $s > 1$. If $\partial\Omega$ is C^∞ and non-characteristic for X, then the first Dirichlet eigenvalue λ_1 of $-\triangle_X$ is positive and we have the following Poincaré inequality*

$$\lambda_1 \|u\|^2_{L^2(\Omega)} \leq \int_\Omega |Xu|^2 dx, \ \forall u \in H^1_{X,0}(\Omega).$$

Proof: We set

$$\lambda_1 = \inf_{\|\varphi\|_{L^2(\Omega)}=1, \varphi \in H^1_{X,0}(\Omega)} \{\|X\varphi\|^2_{L^2(\Omega)}\}.$$

Suppose that $\lambda_1 = 0$. Then there exists $\{\varphi_j\} \subset H^1_{X,0}(\Omega)$ such that $\|X\varphi_j\|_{L^2(\Omega)} \to 0$ and $\|X\varphi_j\|_{L^2(\Omega)} = 1$. Then Proposition 2.2.1 tells us that $H^1_{X,0}(\Omega)$ is compactly embedded into $L^2(\Omega)$. The variational calculus deduces that there exists $\tilde{\varphi} \in H^1_{X,0}(\Omega)$, $\|\tilde{\varphi}\|_{L^2(\Omega)} = 1$, $\tilde{\varphi} \geq 0$ satisfying

$$\triangle_X \tilde{\varphi} = 0, \ \|X\tilde{\varphi}\|_{L^2(\Omega)} = 0.$$

Since X satisfies the logarithmic regularity estimate (2.1.14) with $s > 1$, then $\triangle_X$ is hypoelliptic in Ω, we have $\tilde{\varphi} \in C^\infty(\Omega)$ and

$$X_j \tilde{\varphi}(x) = 0, \ \forall\ x \in \Omega, \ j = 1, \cdots, m.$$

This implies that $\tilde{\varphi}$ is constant along the integral paths of vector fields of $X_1, \cdots, X_m$. Now the controllability of Proposition 2.1.1 implies that $\tilde{\varphi}$ is constant on each connected component of Ω.

Since $\partial\Omega$ is non-characteristic, by taking $x_0 \in \partial\Omega$, then there exists a X_j such that if $X_j\tilde{\varphi} = 0$ we have $\tilde{\varphi}(x) = 0$ near x_0, which means $\tilde{\varphi}(x) = 0$ on Ω. This is impossible because $\|\tilde{\varphi}\|_{L^2(\Omega)} = 1$, so we prove finally $\lambda_1 > 0$. □

2.2.2 Logarithmic Non-linear Case

If the vector fields X satisfies Hörmander's condition with the Hörmander index Q and $\partial\Omega$ is non characteristic for X. We know the Sobolev critical embedding $H^1_{X,0}(\Omega) \hookrightarrow L^{2\bar{\nu}/(\bar{\nu}-2)}(\Omega)$, here the general Métivier index $n + Q - 1 \leq \bar{\nu} \leq nQ$. If X is infinitely degenerate (i.e. $Q \to +\infty$), then we can only expect to get the compactly embedding $H^1_{X,0}(\Omega) \hookrightarrow L^2(\Omega)$ (see Proposition 2.2.1). That means that if the non-linear term of the equation is the power-non-linearity such as u^p with $p > 1$, we can not ensure the existence of nontrivial weak solution in the infinitely degenerate case. Fortunately, by using logarithmic Sobolev inequality (2.2.1), we can consider the following boundary value problem with logarithmic-non-linearity term:

$$\begin{cases} -\triangle_X u = au \log|u| + bu, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{2.2.7}$$

where Ω is a bounded open domain of $\mathbb{R}^n$, a, b are constants, $X = \{X_1, X_2, \cdots, X_m\}$ is C^∞ smooth real vector fields defined on Ω, which is infinitely degenerate on a hypersurface $\Gamma \subset \Omega$ and satisfies the finite type of Hörmander's condition with Hörmander index $Q \geq 1$ on $\Omega \backslash \Gamma$. $\triangle_X = \sum_{j=1}^m X_j^2$ is an infinitely degenerate elliptic operator. Here we assume both $\partial\Omega$ and Γ are C^∞ smooth and non-characteristic for the system of vector fields X.

Theorem 2.2.2. *If $a \neq 0$, X satisfies the logarithmic regularity estimate* (2.1.14) *with $s > 1$.*

(1) *Then the problem* (2.2.7) *possesses at least one nonzero weak solution in $H^1_{X,0}(\Omega)$.*

(2) *Moreover if $a > 0$, Then the problem* (2.2.7) *possesses infinitely many weak solutions in $H^1_{X,0}(\Omega)$.*

For $a \in \mathbb{R}, a \neq 0$, we study now the following variational problems

$$I_a = \inf_{\{u \in H^1_{X,0}(\Omega), \|u\|_{L^2(\Omega)} = 1\}} I_a(u), \tag{2.2.8}$$

with

$$I_a(u) = \|Xu\|^2_{L^2(\Omega)} - a \int_\Omega |u|^2 \log|u| dx.$$

Proposition 2.2.3. *Under the hypothesis of Theorem 2.2.2, I_a is an attained minimum in $H^1_{X,0}(\Omega)$.*

Proposition 2.2.4. *The minimizer u of variational problem* (2.2.8) *is a non trivial weak solution of the following semilinear Dirichlet problem*

$$\begin{cases} -\triangle_X u = au \log|u| + I_a u, & \textit{in } \Omega, \\ u = 0, & \textit{on } \partial\Omega, \end{cases} \tag{2.2.9}$$

Proof of Theorem 2.2.2(1): From Proposition 2.2.3 and Proposition 2.2.4, there exists a weak solution $\tilde{u}$ of (2.2.9). For $c > 0$, we set $u = c\tilde{u}$, then $\|u\|_{L^2(\Omega)} = c > 0, u \in H^1_{X,0}(\Omega)$ and in the weak sense

$$-\triangle_X u = au \log|u| + (I_a - a \log c)u.$$

Choose $c = e^{\frac{I_a - b}{a}} > 0$, then u is a non trivial weak solution of (2.2.7). $\square$

Proof of Proposition 2.2.3: First, we prove that $I_a(v)$ is bounded below on

$$\{u \in H^1_{X,0}(\Omega), \|u\|_{L^2(\Omega)} = 1\}.$$

Logarithmic Sobolev inequality give that

$$\int_\Omega |u|^2 |\log(\frac{|u|}{\|u\|_{L^2(\Omega)}})|^2 dx \le C_0 \Big[\int_\Omega |Xu|^2 dx + \|u\|^2_{L^2(\Omega)}\Big], \ \forall\ u \in H^1_{X,0}(\Omega). \tag{2.2.10}$$

For all $a \neq 0$, we have

$$\begin{aligned} |a \int_\Omega |u|^2 \log |u| dx| &\le \frac{1}{2C_0} \int_\Omega |u|^2 |\log |u||^2 dx + \frac{C_0|a|^2}{2} \\ &\le \frac{1}{2}\|Xu\|^2_{L^2(\Omega)} + \frac{1+C_0|a|^2}{2}, \end{aligned}$$

for all $u \in \{u \in H^1_{X,0}(\Omega), \|u\|_{L^2(\Omega)} = 1\}$. We have that

$$I_a(u) = \|Xu\|^2_{L^2(\Omega)} - |a| \int_\Omega |u|^2 \big|\log |u|\big| dx \ge \frac{\lambda_1 - 1 - C_0|a|^2}{2},$$

for all $u \in \{u \in H^1_{X,0}(\Omega), \|u\|_{L^2(\Omega)} = 1\}$.

Now let $\{u_j\} \subset \{u \in H^1_{X,0}(\Omega), \|u\|_{L^2(\Omega)} = 1\}$ be a minimazer sequence of I_a, then

$$\frac{1+C_0|a|^2}{2} + I_a(u_j) \ge \frac{\|Xu_j\|^2_{L^2(\Omega)}}{2}.$$

It follows that $\{u_j\}$ is a bounded sequence in $H^1_{X,0}(\Omega)$. Then there exists a subsequence (denote still by $\{u_j\}$) such that $u_j \rightharpoonup u_0$ in $H^1_{X,0}(\Omega)$ and $u_j \to u_0$ in $L^2(\Omega)$. Also from $I_a(u) = I_a(|u|)$, we suppose $u_0 \ge 0$,

$$\liminf_{j\to\infty} \|Xu_j\|^2_{L^2(\Omega)} \ge \|Xu_0\|^2_{L^2(\Omega)}, \quad \lim_{j\to\infty} \|u_j\|_{L^2(\Omega)} = \|u_0\|_{L^2(\Omega)} = 1.$$

Then

$$I_a(u_0) \le I_a(u_j), j \to \infty, \ I_a(u_0) \le I_a, \ u_0 \in \{u \in H^1_{X,0}(\Omega), \ \|u\|_{L^2(\Omega)} = 1\}.$$

So I_a is an attained minimum in $H^1_{X,0}(\Omega)$. □

Proof of Proposition 2.2.4: From Proposition 2.2.3, the minimizer

$$u \in \{u \in H^1_{X,0}(\Omega), \|u\|_{L^2(\Omega)} = 1\}$$

is a weak solution of (2.2.9), which is equivalent to

$$\int_\Omega \sum_{j=1}^m X_j u X_j \varphi dx - \int_\Omega a u \varphi \log |u| dx - I_a \int_\Omega u \varphi dx = 0, \tag{2.2.11}$$

for all $\varphi \in H^1_{X,0}(\Omega)$. For fixed $\varphi \in H^1_{X,0}(\Omega)$ and $\mu \in \mathbb{R}$ with $|\mu|$ small enough, we put

$$u_\mu = u + \mu\varphi, \quad \tilde{u}_\mu = \frac{u_\mu}{\|u_\mu\|_{L^2(\Omega)}},$$

then $\tilde{u}_\mu \in \{u \in H^1_{X,0}(\Omega), \|u\|_{L^2(\Omega)} = 1\}$, so that

$$H(\mu) = I_a(\tilde{u}_\mu) \geq I_a(u) = I_a,$$

and

$$H(\mu) = \frac{1}{\|u_\mu\|^2_{L^2(\Omega)}} I_a(u_\mu) + a \log \|u_\mu\|_{L^2(\Omega)}.$$

By direct calculus,

$$\begin{aligned} H'(\mu) = & \frac{1}{\|u_\mu\|^2_{L^2(\Omega)}} (2\int_\Omega X u_\mu X \varphi dx - 2a \int_\Omega u_\mu \varphi \log |u_\mu| dx - a \int_\Omega u_\mu \varphi dx) \\ & - \frac{2}{\|u_\mu\|^4_{L^2(\Omega)}} I_a(u_\mu) \int_\Omega u_\mu \varphi dx + \frac{a}{\|u_\mu\|^2_{L^2(\Omega)}} \int_\Omega u_\mu \varphi dx. \end{aligned}$$

From Lebesgue dominant theorem and using the fact $|t \log t| \leq t^2 + e^{-1}, \forall t \geq 0$, we have

$$\lim_{\mu \to 0} \int_\Omega u_\mu \varphi \log |u_\mu| dx = \int_\Omega u \varphi \log |u| dx.$$

So, $H'(\mu)$ is continuous at $\mu = 0$, then for any $\mu \in \mathbb{R}$, with $|\mu|$ small enough

$$I_a(\tilde{u}_\mu) = H(\mu) = H(0) + H'(0)\mu + \delta(\mu)\mu \geq I_a(u) = H(0),$$

where $\delta(\mu) \to 0$ as $\mu \to 0$. We get finally $H'(0) = 0$, this is true for all $\varphi \in H^1_{X,0}(\Omega)$, we have proved the Proposition 2.2.4. $\square$

Definition 2.2.1. *We say that $u \in H^1_{X,0}(\Omega)$ is a weak solution of* (2.2.7) *if*

$$\int_\Omega \sum_{j=1}^m X_j u X_j v dx - \int_\Omega a u v \log |u| dx - \int_\Omega b u v dx = 0, \ \forall v \in H^1_{X,0}(\Omega). \tag{2.2.12}$$

Now we introduce the following energy functional $E : H^1_{X,0}(\Omega) \to \mathbb{R}$, defined as

$$E(u) = \frac{1}{2}\Big(\int_\Omega \sum_{j=1}^m (X_j u)^2 dx - \int_\Omega a u^2 \log |u| dx + \int_\Omega \frac{a u^2}{2} dx - \int_\Omega b u^2 dx \Big). \tag{2.2.13}$$

From Theorem 2.2.1, we know that, $E(u) \in C^1(H^1_{X,0}(\Omega), \mathbb{R})$. Thus (2.2.7) is the Euler-Lagrange equation of the variational problem for the energy functional (2.2.13), and its Fréchet differentiation is given by

$$\langle E'(u), v \rangle = \int_\Omega \sum_{j=1}^m X_j u X_j v dx - \int_\Omega a u v \log |u| dx - \int_\Omega b u v dx, \ \forall \ u, \ v \in H^1_{X,0}(\Omega). \tag{2.2.14}$$

Thus the critical point of $E(u)$ in $H^1_{X,0}(\Omega)$ is the weak solution of (2.2.7).

Definition 2.2.2 (Palais-Smale Condition)**.** *Let V be a Banach space, $E \in C^1(V; \mathbb{R})$ and $c \in \mathbb{R}$. We say that E satisfies the $(PS)_c$ condition, if for any sequence $\{u_k\} \subset V$ with the properties:*

$$E(u_k) \to c \quad \text{and} \quad \| E'(u_k) \|_{V'} \to 0,$$

there exists a subsequence which is convergent in V, where $E'(\cdot)$ is the Fréchet differentiation of E and V' is the dual space of V. If it holds for any $c \in \mathbb{R}$, we say that E satisfies the (PS) condition.

Proposition 2.2.5 (Mountain Pass Theorem, c.f. [54])**.** *Let V be a Banach space and $E \in C^1(V, \mathbb{R})$. Suppose $E(0) = 0$ and it satisfies (1) there exist $R > 0$ and $\lambda > 0$, such that if $\|u\|_V = R$, then $E(u) \geq \lambda$; (2) there exists $e \in V$, such that $\|e\|_V > R$ and $E(e) < \lambda$.*

If E satisfies the $(PS)_c$ condition with

$$c = \inf_{h\in\chi} \max_{t\in[0,1]} E(h(t)),$$

where

$$\chi = \{h \in C([0,1]; V) \,|\, h(0) = 0 \quad \text{and} \quad h(1) = e\},$$

then c is a critical value of E and $c \geq \lambda$.

Proposition 2.2.6 (Symmetrical Mountain Pass theorem, c.f. [54])**.** *Suppose V is an infinite dimensional Banach space and $E \in C^1(V, \mathbb{R})$ satisfies (PS) condition, $E(u) = E(-u)$ for all u, and $E(0) = 0$. Suppose $V = V^- \bigoplus V^+$, where V^- is finite dimensional, and assume the following conditions,*
(1). $\exists \alpha > 0$, $\rho > 0$, and for any $u \in V^+$, $\|u\| = \rho$, we have $E(u) \geq \alpha$.
(2). For any finite dimensional subspace $W \subset V$, there is $R = R(W)$ such that $E(u) \leq 0$ for $u \in W, \|u\| \geq R$. Then E possesses an unbounded sequence of critical values.

Proposition 2.2.7. *If $a > 0$, there exist $R > 0$ and $\lambda > 0$, such that*
(1). $E(u) \geq \lambda$, for any $\|u\|_{H^1_{X,0}(\Omega)} = R$;
(2). E satisfies (PS) condition.

Proof: First, by using Hölder's inequality, Logarithmic Sobolev inequality and Poincáre inequality, we have

$$\begin{aligned}
E(u) &= \frac{1}{2}\Big(\int_\Omega \sum_{j=1}^m (X_j u)^2 dx - \int_\Omega au^2 \log|u| dx + \int_\Omega \frac{au^2}{2} dx - \int_\Omega bu^2 dx\Big) \\
&= \frac{1}{2}\Big(\|Xu\|^2_{L^2(\Omega)} - a\int_\Omega u^2 \log\frac{|u|}{\|u\|_{L^2(\Omega)}} dx - a\log\|u\|_{L^2(\Omega)} \int_\Omega u^2 dx \\
&\quad + \int_\Omega \frac{au^2}{2} dx - \int_\Omega bu^2 dx\Big) \\
&\geq \frac{1}{2}\Big(\|Xu\|^2_{L^2(\Omega)} - \frac{1}{2C_0}\int_\Omega |u|^2 |\log(\frac{|u|}{\|u\|_{L^2(\Omega)}})|^{2s-1} dx - C_2\int_\Omega u^2 dx \\
&\quad - a\log\|u\|_{L^2(\Omega)} \int_\Omega u^2 dx\Big) \\
&\geq \frac{1}{2}\Big(\frac{\lambda_1}{2(1+\lambda_1)}\|u\|^2_{H^1_{X,0}(\Omega)} - (C_2 + \frac{1}{2})\|u\|^2_{L^2(\Omega)} - a\log\|u\|_{L^2(\Omega)}\int_\Omega u^2 dx\Big),
\end{aligned} \tag{2.2.15}$$

where C_0 and λ_1 are positive constants given by (2.1.14) and (2.1.21), and

$$C_2 = \frac{2s-2}{2s-1}\Big(\frac{2C_0 a^{2s-1}}{2s-1}\Big)^{\frac{1}{2s-2}} + b - \frac{a}{2}. \tag{2.2.16}$$

We set $B_R = \{u \in H^1_{X,0}(\Omega), \|u\|_{H^1_{X,0}(\Omega)} < R\}$, and take $R = \exp\{-(2C_2+1)/(2a)\}$, then

$$E(u)|_{\partial B_R} \geq \lambda_1 R^2 / \big(4(1+\lambda_1)\big).$$

Let $\lambda = \lambda_1 R^2/\big(4(1+\lambda_1)\big) > 0$, then $E(u)|_{\partial B_R} \geq \lambda$. The result of Proposition 2.2.7(1) is proved.

Next, let $c_0 \in \mathbb{R}$, and $\{u_m\} \subset H^1_{X,0}(\Omega)$ satisfy

$$E(u_m) \to c_0, \text{ and } \|J'(u_m)\|_{H_X^{-1}(\Omega)} \to 0.$$

Then we can prove that the (PS) sequence u_m is bounded in $H^1_{X,0}(\Omega)$. Indeed, for m sufficiently large, we obtain, from (2.2.13), that

$$c_1 + o(1)\|u_m\|_{H^1_{X,0}(\Omega)} \geq E(u_m) - \frac{1}{2}\langle E'(u_m), u_m\rangle = \int_\Omega \frac{au_m^2}{4} dx,$$

where $c_1 = c_0 + 1$, which means

$$\int_\Omega u_m^2 dx \leq M_1 + o(1)\|u_m\|_{H^1_{X,0}(\Omega)}, \tag{2.2.17}$$

where $M_1 = \frac{4c_1}{a}$ is independent of m. Next, for m large enough, from (2.2.15), we have

$$c_1 \geq E(u_m) \geq \frac{\lambda_1}{4(1+\lambda_1)}\|u_m\|^2_{H^1_{X,0}(\Omega)} - \frac{2C_2+1}{4}\|u_m\|^2_{L^2(\Omega)} - \frac{a}{2}\log\|u_m\|_{L^2(\Omega)} \int_\Omega u_m^2 dx,$$

where C_2 is the constant in (2.2.16). Since $|t\log t| \leq t^2 + e^{-1}$ for $t \geq 0$, it yields

$$\begin{aligned}
\frac{\lambda_1}{1+\lambda_1}\|u_m\|^2_{H^1_{X,0}(\Omega)} &\leq 4c_1 + (2C_2+1)\|u_m\|^2_{L^2(\Omega)} + a\|u_m\|^2_{L^2(\Omega)}|\log\|u_m\|^2_{L^2(\Omega)}| \\
&\leq 4c_1 + (2C_2+1)\|u_m\|^2_{L^2(\Omega)} + a(\|u_m\|^4_{L^2(\Omega)} + e^{-1}) \\
&\leq (4c_1 + ae^{-1}) + (2C_2+1)\|u_m\|^2_{L^2(\Omega)} + a\|u_m\|^4_{L^2(\Omega)},
\end{aligned}$$

which implies that, combining with (2.2.17),

$$\Big(\frac{\lambda_1}{1+\lambda_1} + o(1)\Big)\|u_m\|^2_{H^1_{X,0}(\Omega)} \leq M_2.$$

This means the sequence $\{u_m\}$ is bounded in $H^1_{X,0}(\Omega)$, as claimed.

Thus we can deduce that there exists a subsequence (still denoted by $\{u_m\}$), such that

$$u_m \rightharpoonup u \text{ in } H^1_{X,0}(\Omega), \text{ and } u_m \to u \text{ in } L^2(\Omega).$$

Now from $\langle J'(u_m), u_m - u\rangle = o(1)$, as $m \to \infty$, we obtain

$$\begin{aligned}
&\lim_{m\to\infty}\Big\{\|Xu_m\|^2_{L^2(\Omega)} - \int_\Omega au_m^2\log|u_m|dx - \int_\Omega bu_m^2 dx\Big\} \\
&= \|Xu\|^2_{L^2(\Omega)} - \int_\Omega au^2\log|u|dx - \int_\Omega bu^2 dx.
\end{aligned}$$

By the results of Lemma 2.2.1, one has

$$\lim_{m\to\infty}\int_\Omega au_m^2\log|u_m|dx = \int_\Omega au^2\log|u|dx.$$

This means $u_m \to u$ strongly in $H^1_{X,0}(\Omega)$. So $E(u)$ satisfies (PS) condition. Proposition 2.2.7 is proved. □

Proof of Theorem 2.2.2(2): Due to Proposition 2.3.2, we know that the operator $-\triangle_X$ has a sequence of discrete eigenvalues $0 < \lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots \le \lambda_k \le \cdots$, with $\lambda_k \to +\infty$, and the corresponding eigenfunction is denoted by $\{\varphi_k\}$ which is an orthonormal basis of $H^1_{X,0}(\Omega)$.

Now, we take $u \in V = H^1_{X,0}(\Omega)$, then $E(u) = E(-u)$ and $E(0) = 0$. Taking $k_0 \ge 1$, $V^+_{k_0}$ =span$\{\varphi_k; k \ge k_0 + 1\}$ and $V^-_{k_0} =$ span$\{\varphi_k; k \le k_0\}$, we have $V = V^-_{k_0} \bigoplus V^+_{k_0}$. Similar to the proof of Proposition 2.2.7(1), we can deduce that there exist $\rho > 0$ and $\alpha > 0$, such that for any $u \in V^+_{k_0}$ with $\|u\|_{H^1_{X,0}}(\Omega) = \rho$, we have $E(u) \ge \alpha > 0$, the condition (1) of Proposition 2.2.6 holds.

On the other hand, for any finite dimensional subspace $W \subset H^1_{X,0}(\Omega)$, we know that there exists $k_0 \ge 1$, such that $W \subset V^-_{k_0}$=span $\{\varphi_k; k \le k_0\}$. Thus there holds for any $w \in W$ and $0 < \varepsilon < 1$,

$$\int_\Omega (Xw)^2 dx \le \lambda_{k_0} \int_\Omega w^2 dx \le \lambda_{k_0} \|w\|^2_{H^1_{X,0}(\Omega)}.$$

For any nonzero $u \in W$, we take $t > 0$, then

$$\begin{aligned} E(tu) &= t^2 \int_\Omega |Xu|^2 dx - t^2 \int_\Omega au^2 \log|tu| dx + t^2 \int_\Omega \frac{au^2}{2} dx - t^2 \int_\Omega bu^2 dx \\ &\le \lambda_{k_0} t^2 \|u\|^2_{H^1_{X,0}(\Omega)} - t^2 \Big[\log|t| \int_\Omega au^2 dx - \int_\Omega au^2 \log|u| dx + \int_\Omega \frac{au^2}{2} dx - \int_\Omega bu^2 dx \Big]. \end{aligned}$$

Thus for $R = R(W) > 0$ and any nonzero $u \in W$, we take $t > 0$ large enough, then there exist positive constants C_1 and C_2, such that

$$\sup_{\{u \in W, \|tu\|_{H^1_{X,0}(\Omega)} \ge R\}} E(tu) < C_1 t^2 - C_2 t^2 \log|t| \to -\infty, \text{ as } t \to +\infty.$$

This means the condition (2) of Proposition 2.2.6 is satisfied. Hence the functional J has a unbounded sequence of critical values. Actually, Proposition 2.2.6 guarantees the existence of following unbounded sequences of critical values for the functional E,

$$\beta_k = \inf_{u \in \chi_k} \sup_{u \in W_k} E(h(u)), \quad \text{for } k \ge k_0, \tag{2.2.18}$$

here $W_k = \text{span}\{\varphi_j; j \le k\}$, and

$$\begin{aligned} \chi_k = \Big\{ h \in C^0(H^1_{X,0}(\Omega); H^1_{X,0}(\Omega));\ h \text{ is odd }, h(u) = u \text{ if } u \in W_j \\ \text{and } \|u\|_{H^1_{X,0}(\Omega)} \ge R_j \text{ for } j \le k \text{ and } R_j > 0 \Big\}. \end{aligned}$$

Therefore, there exists a non-trivial sequence $u_k \in H^1_{X,0}(\Omega)$ satisfying

$$E(u_k) = \beta_k, \quad \text{and } \langle E'(u_k), v \rangle = 0 \text{ for any } v \in H^1_{X,0}(\Omega).$$

Hence, (2.2.7) possesses infinitely many non-trivial weak solutions. □

Next, we study the following boundary value problem of semi-linear infinitely degenerate elliptic equation with potential term:

$$\begin{cases} -\triangle_X u - \varepsilon V_n u = au \log|u| + bu & \text{in } \Omega\ , \\ u = 0 & \text{on } \partial\Omega\ , \end{cases} \tag{2.2.19}$$

where Ω is a bounded open domain of $\mathbb{R}^n$, a, b are constants and $X = \{X_1, X_2, \cdots, X_m\}$ is C^∞ smooth real vector fields defined on Ω, which .

Now, we consider following conditions:

(H-1) $\partial\Omega$ is C^∞ and non characteristic for the system of vector fields X;
(H-2) X is infinitely degenerate on a non-characteristic hypersurface $\Gamma \subset \Omega$ and satisfies the finite type of Hörmander's condition with Hörmander index $Q \geq 1$ on $\Omega \backslash \Gamma$;
(H-3) X satisfies Logarithmic regularity estimate (2.1.14) with $s \geq 3/2$;
(H-4) The non-negative singular potential function $V_n(x) \in C^\infty(\Omega \setminus \{0\})$ is unbounded at $\{0, 0, \cdots, 0\} \in \Gamma$, and satisfies the Hardy inequality

$$\int_\Omega V_n u^2 dx \leq \int_\Omega |Xu|^2 dx, \text{ for all } u \in H^1_{X,0}(\Omega). \tag{2.2.20}$$

To study the existence and regularity of the solution to (2.2.19), we first give examples satisfying the Hardy inequality.

Proposition 2.2.8. *Let* $X = (\partial_{x_1}, \cdots, \partial_{x_{n-1}}, \varphi(x')\partial_{x_n})$, *where*

$$\varphi(x') = \begin{cases} e^{-\frac{1}{|x_1|^{1/s}}}, & x_1 \neq 0, \\ 0, & x_1 = 0, \end{cases}$$

with $s > 1$, $x' = (x_1, x_2, \cdots, x_{n-1})$.

(1) *If* $V_{n,1}(x) = (\frac{n-3}{2})^2 \frac{1}{|x|^2}$, *then* $V_{n,1}(x) \in C^\infty(\Omega \backslash \{0\})$ *(for* $n \geq 3$*), and*

$$\int_\Omega V_{n,1} u^2 dx \leq \int_\Omega |Xu|^2 dx, \quad \text{for any } u \in H^1_{X,0}(\Omega). \tag{2.2.21}$$

(2) *If* $V_{n,2}(x) = (\frac{n-2}{2})^2 \frac{x_1^{-2} \exp(-\frac{1}{|x_1|^2})}{\exp(-\frac{1}{|x_1|^2}) + \sum_{i=2}^n x_i^2}$, $x = (x_1, x'') = (x_1, x_2, \cdots, x_n)$, *then* $V_{n,2}(x) \in C^\infty(\Omega \backslash \{0\})$ *(for* $n \geq 3$*), and when* $x_1 \to 0$ *we have* $V_{n,2}(x_1, x'') \to 0$ *if* $x'' \neq 0$ *and* $V_{n,2}(x_1, x'') \to +\infty$ *if* $x'' = 0$. *Thus for* $\Omega \subset \left\{x = (x_1, x'') \in \mathbb{R}^n, |\ |x_1| \leq \sqrt{\frac{1}{5}}\right\}$, *there holds*

$$\int_\Omega V_{n,2} u^2 dx \leq \int_\Omega |Xu|^2 dx, \quad \text{for any } u \in H^1_{X,0}(\Omega). \tag{2.2.22}$$

Lemma 2.2.2. *For* $n \geq 3$, $C_0^\infty(\Omega \backslash \{0\})$ *is dense in* $H^1_{X,0}(\Omega)$.

Proof of Proposition 2.2.8: From Lemma 2.2.2, we only need to prove the results for the function $u \in C_0^\infty(\Omega \backslash \{0\})$.

(1). Take a radial vector field R_1 as,

$$R_1 = x_1 \partial x_1 + x_2 \partial x_2 + \cdots + x_{n-1} \partial x_{n-1} + x_n \varphi(x') \partial x_n,$$

then one has $R(V_{n,1}) \geq -2V_{n,1}$ and $div(R_1) = n - 1 + \varphi(x')$. Thus

$$\int_\Omega -2V_{n,1} u^2 dx \leq \int_\Omega R_1(V_{n,1}) u^2 dx = -\int_\Omega div(R_1) V_{n,1} u^2 dx - \int_\Omega V_{n,1} R_1(u^2) dx.$$

This implies

$$\int_\Omega (n - 3 + \varphi(x')) V_{n,1} u^2 dx \leq -\int_\Omega V_{n,1} R_1(u^2) dx, \tag{2.2.23}$$

and

$$\begin{aligned}
&-\int_\Omega V_{n,1}R_1(u^2)dx = -2\int_\Omega V_{n,1}uR_1(u)dx \\
&= -\int_\Omega V_{n,1}(2ux_1\partial_{x_1}u + 2ux_2\partial_{x_2}u + \cdots + 2ux_{n-1}\partial_{x_{n-1}}u + 2ux_n\varphi(x')\partial_{x_n}u)dx \\
&\le 2\Big(\int_\Omega V_{n,1}^2(x_1^2+x_2^2+\cdots+x_n^2)u^2dx\Big)^{\frac{1}{2}}\Big(\int_\Omega(\Sigma_{i=1}^{n-1}(\partial_{x_i}u)^2+(\varphi(x')\partial_{x_n}u)^2)dx\Big)^{\frac{1}{2}}.
\end{aligned}$$

Observe that,

$$V_{n,1}(x_1^2+x_2^2+\cdots+x_n^2) = (\frac{n-3}{2})^2,$$

and

$$n-3+\varphi(x') \ge n-3.$$

Then we deduce from (2.2.23) that,

$$\int_\Omega V_{n,1}u^2dx \le \Big(\int_\Omega V_{n,1}u^2dx\Big)^{\frac{1}{2}}\Big(\int_\Omega |Xu|^2dx\Big)^{\frac{1}{2}},$$

which means

$$\int_\Omega V_{n,1}u^2dx \le \int_\Omega |Xu|^2dx,$$

as claimed.

(2). For $V_{n,2}$, we take the following radial vector field R_2,

$$R_2 = x_1^3\partial x_1 + x_2\partial x_2 + \cdots + x_{n-1}\partial x_{n-1} + x_n\varphi(x')\partial x_n,$$

then $R_2(V_{n,2}) \ge -2x_1^2V_{n,2}$ and $div(R_2) = 3x_1^2+n-2+\varphi(x')$, which means

$$\int_\Omega -2x_1^2V_nu^2dx \le \int_\Omega R_2(V_{n,2})u^2dx = -\int_\Omega div(R_2)V_{n,2}u^2dx - \int_\Omega V_{n,2}R_2(u^2)dx.$$

Thus we have

$$\int_\Omega (x_1^2+n-2+\varphi(x'))V_{n,2}u^2dx \le -\int_\Omega V_{n,2}R_2(u^2)dx, \tag{2.2.24}$$

and

$$\begin{aligned}
&-\int_\Omega V_{n,2}R_2(u^2)dx = -2\int_\Omega V_{n,2}uR_2(u)dx \\
&= -\int_\Omega V_{n,2}(2ux_1^3\partial_{x_1}u + 2ux_2\partial_{x_2}u + \cdots + 2ux_{n-1}\partial_{x_{n-1}}u + 2ux_n\varphi(x')\partial_{x_n}u)dx \\
&\le 2\Big(\int_\Omega V_{n,2}^2(x_1^6+x_2^2+\cdots+x_n^2)u^2dx\Big)^{\frac{1}{2}}\Big(\int_\Omega(\Sigma_{i=1}^{n-1}(\partial_{x_i}u)^2+(\varphi(x')\partial_{x_n}u)^2)dx\Big)^{\frac{1}{2}}.
\end{aligned}$$

Since $x_1^6 \ge \exp\big\{-\frac{1}{|x_1|^2}\big\}$ for $|x_1| \le \sqrt{\frac{1}{5}}$, then

$$V_{n,2}(x_1^6+x_2^2+\cdots+x_n^2) \le x_1^4(\frac{n-2}{2})^2 \le (\frac{n-2}{2})^2,$$

and

$$x_1^2+n-2+\varphi(x') \ge n-2.$$

Thus we have from (2.2.24),

$$\int_\Omega V_{n,2}u^2dx \le (\int_\Omega V_{n,2}u^2dx)^{\frac{1}{2}}(\int_\Omega |Xu|^2dx)^{\frac{1}{2}},$$

which implies

$$\int_\Omega V_{n,2}u^2dx \le \int_\Omega |Xu|^2dx.$$

Proposition 2.2.8 is proved. □

Proof of Lemma 2.2.2: By the definition of $H^1_{X,0}(\Omega)$, it suffices to show that

$$C_0^\infty(\Omega) \subset \overline{C_0^\infty(\Omega\backslash\{0\})}^{\|\cdot\|_{H^1_{X,0}}}.$$

Let ϕ be a C^∞ function, satisfying

$$\phi(\eta) = \begin{cases} 0 & \text{if } 0 < \eta \le 1, \\ 1 & \text{if } \eta \ge 2. \end{cases}$$

For $u \in C_0^\infty(\Omega)$, let $\varepsilon > 0$ small enough, and then we set $u_\varepsilon(x) = \phi\big(\frac{1}{\varepsilon}|x|\big)u(x)$. Thus $u_\varepsilon(x) \in C_0^\infty(\Omega\backslash\{0\})$ and

$$\|u_\varepsilon - u\|^2_{H^1_{X,0}(\Omega)} = \|X(u_\varepsilon - u)\|^2_{L^2(\Omega)} + \|u_\varepsilon - u\|^2_{L^2(\Omega)}.$$

By using the dominated convergence theorem we have that, as $\varepsilon \to 0$,

$$\|u_\varepsilon - u\|^2_{L^2(\Omega)} \to 0, \text{ and } \int_\Omega |\phi\big(\tfrac{1}{\varepsilon}|x|\big) - 1|^2|Xu(x)|^2dx \to 0.$$

On the other hand, we know that

$$\begin{aligned}
&\int_\Omega |X\big(\frac{1}{\varepsilon}|x|\big)|^2|\nabla\phi\big(\frac{1}{\varepsilon}|x|\big)|^2|u(x)|^2dx \\
&\le \frac{C}{\varepsilon^2}\int_\Omega |\nabla\phi\big(\frac{1}{\varepsilon}|x|\big)|^2|u(x)|^2dx \\
&\le \frac{C}{\varepsilon^2}\|u\|^2_{L^\infty}\|\nabla\phi\|^2_{L^\infty(\Omega)}\int_{\{\varepsilon\le|x|\le2\varepsilon\}}dx \\
&\le C'\varepsilon^{n-2} \to 0, \text{ as } \varepsilon \to 0.
\end{aligned}$$

□

Next, we have the following results for existence of solutions to (2.2.19) (also see [8]-[9]).

Theorem 2.2.3. *Under the conditions above, then we have*

(1) *The semi-linear Dirichlet problem* (2.2.19) *possesses at least one nonzero weak solution in* $H^1_{X,0}(\Omega)$.

(2) *Moreover if* $a > 0$, *the semi-linear Dirichlet problem* (2.2.19) *possesses infinitely many weak solutions in* $H^1_{X,0}(\Omega)$.

Remark 2.2.1. *The proof of Theorem 2.2.3 is similar to the proof of Theorem 2.2.2. Also, we need following lemmas mainly concerning the Hardy term* $V_n(x)$.

Lemma 2.2.3. *Under the hypothesis of Theorem 2.2.3, the first eigenvalue η_1 of the operator $-\Delta_X - \varepsilon V_n$ is strictly positive and satisfies the following inequality.*

$$\|u\|^2_{L^2(\Omega)} \le \frac{1}{\eta_1}\Big(\int_\Omega |Xu|^2 dx - \varepsilon \int_\Omega V_n u^2 dx\Big), \qquad \forall u \in H^1_{X,0}(\Omega). \tag{2.2.25}$$

Lemma 2.2.4. *Under the hypotheses of Theorem 2.2.3, the positive operator $-\triangle_X - \varepsilon V_n$ has a sequence of discrete eigenvalues $0 < \eta_1 \le \eta_2 \le \eta_3 \le \cdots \le \eta_k \le \cdots$, and $\eta_k \to \infty$, such that for any $k \ge 1$, the Dirichlet problem*

$$\begin{cases} -\triangle_X \varphi_k - \varepsilon V_n \varphi_k = \eta_k \varphi_k, & \text{in } \Omega, \\ \varphi_k = 0, & \text{on } \partial\Omega, \end{cases} \tag{2.2.26}$$

admits a non trivial solution $\varphi_k \in H^1_{X,0}(\Omega)$. Moreover, $\{\varphi_k\}_{k\ge1}$ constitute an orthonormal basis of the Sobolev space $H^1_{X,0}(\Omega)$.

Lemma 2.2.5. *Let $V_n \in C^\infty(\Omega\backslash\{0\})$ and satisfies the Hardy inequality (2.2.20), $u_m \rightharpoonup u$ in $H^1_{X,0}(\Omega)$, as $m \to +\infty$, then there exists a subsequence $\{u_{m_k}\}$, such that*

(1) $\displaystyle\lim_{k\to\infty}\int_\Omega V_n u_{m_k}\varphi dx = \int_\Omega V_n u \varphi dx$ *for all* $\varphi \in H^1_{X,0}(\Omega)$.

(2) $\displaystyle\lim_{k\to\infty}\int_\Omega V_n u^2_{m_k} dx = \int_\Omega V_n u^2 dx$.

Now, we concern the regularity of the solution to (2.2.19).

Theorem 2.2.4. *Under the conditions* (H-1), (H-2), (H-3) *and* (H-4), *if $0 < \varepsilon < 1$, and $a \ne 0$, then we have*

(1) *If $u_\varepsilon \in H^1_{X,0}(\Omega)$, $u_\varepsilon \ge 0$, and $\|u_\varepsilon\|_{L^2(\Omega)} \ne 0$ is a weak solution of* (2.2.19), *then for $1 < p < \frac{1+\sqrt{1-\varepsilon}}{\varepsilon}$, one has $u_\varepsilon \in L^{2p}(\Omega)$.*

(2) *If $\varepsilon \in (0, \frac{4}{\bar\nu}(1-\frac{1}{\bar\nu}))$, $u_\varepsilon \in H^1_{X,0}(\Omega)$, $u_\varepsilon \ge 0$ and $\|u_\varepsilon\|_{L^2(\Omega)} \ne 0$, is a weak solution of* (2.2.19), *moreover $a < 0$, then $u_\varepsilon \in C^\infty(\Omega \setminus \Gamma) \cap C^0(\overline{\Omega} \setminus \Gamma)$ and $u_\varepsilon(x) > 0$ for all $x \in \Omega \setminus \Gamma$, where $\bar\nu$ is the general Métivier index of X on $\Omega\backslash\Gamma$.*

In case of $1 \le p < \frac{1+\sqrt{1-\varepsilon}}{\varepsilon}$, one has $\frac{2p-1}{p^2} > \varepsilon$. So if p_1 satisfies $\frac{2p_1-1}{p_1^2} > \varepsilon$, we can find a constant $\eta > 0$ such that $\frac{2p_1-1}{p_1^2} = \varepsilon + \eta$, and for $p \in [1, p_1]$, we have $\frac{2p-1}{p^2} \ge \varepsilon + \eta$.

Proposition 2.2.9. *Under the conditions* (H-1), (H-2), (H-3) *and* (H-4), *if $p_0 \in [1, p_1]$, $u \in H^1_{X,0}(\Omega)$ is a weak solution of* (2.2.19), *and $u \ge 0$, $\|u\|_{L^{2p_0}(\Omega)} \ne 0$. Then there exists a constant A_0, such that $\|u\|_{L^{2p_0}(\Omega)} \le A_0$, and for $\tilde u = \frac{u}{\|u\|_{L^{2p_0}}}$, $N = [\frac{1}{\eta}] + 1$, we have ,*

$$\begin{aligned}\int_\Omega |X\tilde u^{p_0}|^2 dx + \int_\Omega \tilde u^{2p_0}\log^2(\tilde u^{p_0})dx \le (N+1)(|a|^2 + 2p_0|b| + 2p_0|a\log A_0|) \\ + (N+2)C_N,\end{aligned} \tag{2.2.27}$$

where $C_N > 0$ depending on N.

Proof: Since $\tilde u \in H^1_{X,0}(\Omega)$ and $\|\tilde u\|_{L^{2p_0}(\Omega)} = 1$, for $p_0 \in [1, p_1]$, then

$$-\Delta_X \tilde u - \varepsilon V_n \tilde u = a\tilde u \log \tilde u + (b + a \log \|u\|_{L^{2p_0}(\Omega)})\tilde u. \tag{2.2.28}$$

Taking $\tilde{u}^{2p_0-1}$ as a test function, we obtain

$$\frac{2p_0-1}{p_0^2}\int_\Omega |X\tilde{u}^{p_0}|^2dx - \varepsilon\int_\Omega V_n\tilde{u}^{2p_0}dx = (b + a\log\|u\|_{L^{2p_0}(\Omega)})\int_\Omega \tilde{u}^{2p_0}dx$$
$$+\frac{a}{p_0}\int_\Omega \tilde{u}^{2p_0}\log\tilde{u}^{p_0}dx,$$

as $\frac{2p_0-1}{p_0^2}\geq \eta+\varepsilon$, one has

$$\eta p_0\int_\Omega |X\tilde{u}^{p_0}|^2dx \leq \frac{1}{2}\int_\Omega \tilde{u}^{2p_0}\log^2\tilde{u}^{p_0}dx + (\frac{1}{2}|a|^2+p_0|b|+p_0|a\log A_0|).$$

Take $N=[\frac{1}{\eta}]+1$, then $\frac{1}{N}<\eta\leq p_0\eta$. So we have

$$\int_\Omega |X\tilde{u}^{p_0}|^2dx \leq \frac{N}{2}\int_\Omega \tilde{u}^{2p_0}\log^2\tilde{u}^{p_0}dx + N(\frac{1}{2}|a|^2+p_0|b|+p_0|a\log A_0|). \tag{2.2.29}$$

By Hölder's inequality and the Logarithmic Sobolev inequality, we know for $N\geq 1$, there is a constant $C_N>0$, such that

$$\int_\Omega \tilde{u}^{2p_0}\log^2\tilde{u}^{p_0}dx \leq \frac{1}{N}\|X(\tilde{u}^{p_0})\|^2_{L^2(\Omega)} + C_N. \tag{2.2.30}$$

By (2.2.29) $\times\frac{2(N+1)}{N}$+(2.2.30)$\times(N+2)$, we get

$$\int_\Omega |X\tilde{u}^{p_0}|^2dx + \int_\Omega \tilde{u}^{2p_0}\log^2(\tilde{u}^{p_0})dx \leq (N+1)(|a|^2+2p_0|b|+2p_0|a\log A_0|)+(N+2)C_N.$$

Proposition 2.2.9 is proved. □

Furthermore, we gain

Proposition 2.2.10. *For $p_0\in[1,p_1]$, we have for any $m\in\mathbb{N}$,*

$$\int_\Omega |X\tilde{u}^{p_0}|^2\log^{2m-2}(\tilde{u}^{p_0})dx + \int_\Omega \tilde{u}^{2p_0}\log^{2m}(\tilde{u}^{p_0})dx \leq M_1^{2m}P(m,p_0)(m!)^2, \tag{2.2.31}$$

where $N=[\frac{1}{\eta}]+1$, $P(m,p_0)=p_0^m$ if $m\leq\sqrt{p_0}$, $P(m,p_0)=p_0^{\sqrt{p_0}}$ if $m>\sqrt{p_0}$, and

$$M_1\geq\Big[163N^2+9N^2C_N+3N(C_{2N}+|\Omega|)+14N^2(|a|^2+2|b|+2|a\log A_0|)\Big]^{\frac{1}{2}}.$$

Proof: From Proposition 2.2.9, the estimate (2.2.31) holds for $m=1$. By induction, we assume that (2.2.31) is hold for $m\in\mathbb{N}$, then we need to prove that (2.2.31) is hold for $m+1$. First let us simplify the notations here, i.e, the notations u, $\tilde{u}$ and p_0 would be denoted by v, u and p respectively, then we take $u^{2p-1}\log^{2m}(u^p)$ as the test function in both sides of the equation (2.2.28) to obtain

$$\frac{2p-1}{p^2}\int_\Omega |Xu^p|^2\log^{2m}(u^p)dx + \frac{2m}{p}\int_\Omega |Xu^p|^2\log^{2m-1}(u^p)dx - \varepsilon\int_\Omega V_n(u^p\log^m(u^p))^2dx$$
$$=\frac{a}{p}\int_\Omega u^{2p}\log^{2m+1}(u^p)dx + (b+a\log\|v\|_{L^{2p}(\Omega)})\int_\Omega u^{2p}\log^{2m}(u^p)dx.$$

By Hardy inequality, we have

$$
\begin{aligned}
\varepsilon \int_\Omega V_n(u^p \log^m(u^p))^2 dx &\leq \varepsilon \int_\Omega |X(u^p \log^m(u^p))|^2 dx \\
&\leq \varepsilon \int_\Omega |Xu^p|^2 \log^{2m}(u^p) dx + 2m\varepsilon \int_\Omega |Xu^p|^2 \log^{2m-1}(u^p) dx \\
&\quad + \varepsilon \int_\Omega |m(Xu^p)\log^{m-1}(u^p)|^2 dx,
\end{aligned}
$$

that means

$$
\begin{aligned}
&p\eta \int_\Omega |Xu^p|^2 \log^{2m}(u^p) dx + 2m \int_\Omega |Xu^p|^2 \log^{2m-1}(u^p) dx - 2pm\varepsilon \int_\Omega |Xu^p|^2 \log^{2m-1}(u^p) dx \\
&\leq |a| \int_\Omega u^{2p} \log^{2m+1}(u^p) dx + p(|b| + |a \log \|v\|_{L^{2p}(\Omega)}|) \int_\Omega u^{2p} \log^{2m}(u^p) dx \\
&+ p\varepsilon \int_\Omega |m(Xu^p)\log^{m-1}(u^p)|^2 dx.
\end{aligned}
$$

Since $\frac{1}{N} < \eta \leq p\eta$, which implies that

$$
\begin{aligned}
&\frac{1}{N} \int_\Omega |Xu^p|^2 \log^{2m}(u^p) dx \\
&\leq |a| \int_\Omega u^{2p} \log^{2m+1}(u^p) dx + p(|b| + |a \log A_0|) \int_\Omega u^{2p} \log^{2m}(u^p) dx \\
&\quad + p\varepsilon \int_\Omega |m(Xu^p)\log^{m-1}(u^p)|^2 dx + (2pm\varepsilon - 2m) \int_\Omega |Xu^p|^2 \log^{2m-1}(u^p) dx.
\end{aligned}
$$

Since $p\varepsilon < \frac{2p-1}{p} < 2$, then $2pm\varepsilon - 2m \leq 2(pm\varepsilon + m) < 6m$, and by Hölder's inequality, one has

$$
\begin{aligned}
&\frac{1}{N} \int_\Omega |Xu^p|^2 \log^{2m}(u^p) dx \\
&\leq \frac{1}{2N} \int_\Omega |Xu^p|^2 |\log^{2m}(u^p))| dx + 20Nm^2 \int_\Omega |Xu^p|^2 \log^{2m-2}(u^p) dx \\
&\quad + \frac{1}{4} \int_\Omega u^{2p} \log^{2m+2}(u^p) dx + (|a|^2 + p|b| + p|a \log A_0|) \int_\Omega u^{2p} \log^{2m}(u^p) dx.
\end{aligned}
$$

Thus

$$
\begin{aligned}
\int_\Omega |Xu^p|^2 \log^{2m}(u^p) dx \leq 40N^2m^2 \int_\Omega |Xu^p|^2 \log^{2m-2}(u^p) dx + \frac{N}{2} \int_\Omega u^{2p} \log^{2m+2}(u^p) dx \\
+ 2N(|a|^2 + p|b| + p|a \log A_0|) M_1^{2m} P(m,p)(m!)^2,
\end{aligned}
$$

which means

$$
\begin{aligned}
&\int_\Omega |Xu^p|^2 \log^{2m}(u^p) dx \\
&\leq 40N^2(m+1)^2 + 2N(|a|^2 + p|b| + p|a \log A_0|) M_1^{2m} P(m,p)(m!)^2 \\
&\quad + \frac{N}{2} \int_\Omega u^{2p} \log^{2m+2}(u^p) dx.
\end{aligned}
\tag{2.2.32}
$$

Now we estimate $\int_\Omega u^{2p}\log^{2m+2}(u^p)dx$. We set $\Omega = \Omega_1 \cup \Omega_2^+ \cup \Omega_2^-$ with $\Omega_1 = \{x \in \Omega; u(x) \leq 1\}$ and

$$\begin{aligned}
\Omega_2^+ &= \{x \in \Omega; u(x) > 1, |\log^m(u(x)^p)| \leq \|u^p \log^m(u^p)\|_{L^2(\Omega)}\},\\
\Omega_2^- &= \{x \in \Omega; u(x) > 1, |\log^m(u(x)^p)| > \|u^p \log^m(u^p)\|_{L^2(\Omega)}\}.
\end{aligned}$$

Then

$$\int_{\Omega_1} u^{2p}\log^{2m+2}(u^p) \leq |\Omega|((m+1)!)^2.$$

Secondly, the estimate (2.2.27) asserts

$$\begin{aligned}
&\int_{\Omega_2^+} u^{2p}\log^{2m+2}(u^p)dx\\
&\leq \|u^p\log^m(u^p)\|_{L^2}^2 \int_\Omega u^{2p}\log^2(u^p)dx\\
&\leq ((N+1)(|a|^2+p|b|+p|a\log A_0|)+(N+2)C_N)M_1^{2m}P(m,p)(m!)^2.
\end{aligned}$$

Next, we estimate the third term. By using the Logarithmic Sobolev inequality, we obtain

$$\begin{aligned}
\int_{\Omega_2^-} u^{2p}\log^{2m+2}(u^p)dx &\leq \int_{\Omega_2^-} (u^p\log^m(u^p))^2 \log^2\Big(\frac{u^p\log^m(u^p)}{\|u^p\log^m(u^p)\|_{L^2(\Omega)}}\Big)dx\\
&\leq \frac{1}{2N}\|X(u^p\log^m(u^p))\|_{L^2(\Omega)}^2 + C_{2N}\|u^p\log^m(u^p)\|_{L^2(\Omega)}^2\\
&\leq \frac{1}{N}\int_\Omega |X(u^p)|^2\log^{2m}(u^p)dx + \frac{m^2}{N}\int_\Omega |X(u^p)|^2\log^{2m-2}(u^p)dx\\
&\quad + C_{2N}\|u^p\log^m(u^p)\|_{L^2(\Omega)}^2\\
&\leq \frac{1}{N}\int_\Omega |X(u^p)|^2\log^{2m}(u^p)dx + (C_{2N}+m^2)M_1^{2m}P(m,p)(m!)^2.
\end{aligned}$$

This implies,

$$\begin{aligned}
&\int_\Omega u^{2p}\log^{2m+2}(u^p)dx\\
&\leq \frac{1}{N}\int_\Omega |X(u^p)|^2\log^{2m}(u^p)dx + |\Omega|((m+1)!)^2 + \Big[(N+2)C_N + C_{2N}\\
&\quad + (N+1)(|a|^2+p|b|+p|a\log A_0|) + m^2\Big]M_1^{2m}P(m,p)(m!)^2.
\end{aligned}\tag{2.2.33}$$

By (2.2.32) $\times\frac{2(N+1)}{N}$+(2.2.33)$\times(N+2)$, and using the facts that $\frac{2(N+1)}{N} \leq 4$, and $N+2 \leq 3N$, we can deduce that

$$\begin{aligned}
&\int_\Omega u^{2p}\log^{2m+2}(u^p)dx + \int_\Omega |Xu^p|^2\log^{2m}(u^p)dx\\
&\leq \Big[163N^2 + 9N^2C_N + 3N(C_{2N}+|\Omega|) + 14N^2(|a|^2+2|b|+2|a\log A_0|)\Big]M_1^{2m}\\
&\quad \cdot P(m+1,p)((m+1)!)^2.
\end{aligned}$$

And this means that if we take

$$M_1 \geq \Big[163N^2 + 9N^2C_N + 3N(C_{2N}+|\Omega|) + 14N^2(|a|^2+2|b|+2|a\log A_0|)\Big]^{\frac{1}{2}},$$

then the estimate (2.2.31) holds for $m+1$. Proposition 2.2.10 is proved. □

Proposition 2.2.11. *Under the conditions of Proposition 2.2.9, if for some* $p_0 \in [1, p_1]$*, there exists* $A_0 \geq e^{12}$*, such that* $\|u\|_{L^{2p_0}(\Omega)} \leq A_0$*, then for* $\tilde{u} = \frac{u}{\|u\|_{L^{2p_0}(\Omega)}}$ *and* $\delta = \frac{1}{2M_1}$*, we have*

$$\int_\Omega u^{2p_0(1+\delta)}dx \leq A_0^{2p_0(1+\delta)(1+(\frac{1}{p_0(1+\delta)})^{\frac{1}{3}})}, \tag{2.2.34}$$

where

$$M_1 \geq \Big[163N^2 + 9N^2C_N + 3N(C_{2N} + |\Omega|) + 14N^2(a^2 + 2b + 2a|\log A_0|)\Big]^{\frac{1}{2}}.$$

Proof: For any $\delta > 0$, one has

$$\begin{aligned}
\Big(\int_\Omega |\tilde{u}^{p_0(1+\delta)}|^2 dx\Big)^{\frac{1}{2}} &= \Big(\int_\Omega |\tilde{u}^{p_0}\tilde{u}^{\delta p_0}|^2 dx\Big)^{\frac{1}{2}} = \Big(\int_\Omega |\tilde{u}^{p_0} e^{\delta \log(\tilde{u}^{p_0})}|^2 dx\Big)^{\frac{1}{2}} \\
&= \Big(\int_\Omega |\tilde{u}^{p_0}\sum_{m=0}^{\infty}\frac{(\delta\log(\tilde{u}^{p_0}))^m}{m!}|^2 dx\Big)^{\frac{1}{2}} \leq \sum_{m=0}^{\infty}\Big(\int_\Omega |\tilde{u}^{p_0}\frac{(\delta\log(\tilde{u}^{p_0}))^m}{m!}|^2 dx\Big)^{\frac{1}{2}} \\
&= \sum_{m=0}^{\infty}\frac{\delta^m}{m!}(\int_\Omega \tilde{u}^{2p_0}\log^{2m}(\tilde{u}^{p_0})dx)^{\frac{1}{2}} \leq \sum_{m=0}^{\infty}\delta^m M_1^m P(m,p_0) \leq p_0^{\sqrt{p_0}}\sum_{m=0}^{\infty}(\delta M_1)^m.
\end{aligned}$$

If $\delta = \frac{1}{2M_1}$, we have

$$\int_\Omega u^{2p_0(1+\delta)}dx \leq 4p_0^{2\sqrt{p_0}}A_0^{2p_0(1+\delta)}.$$

Also for any $p_0 \geq 1$,

$$4p_0^{2\sqrt{p_0}} = 4e^{2\sqrt{p_0}\log p_0} \leq (e^{12})^{2p_0^{\frac{2}{3}}},$$

which implies that if $A_0 \geq e^{12}$, then

$$\int_\Omega u^{2p_0(1+\delta)}dx \leq A_0^{2p_0(1+\delta)(1+(\frac{1}{p_0(1+\delta)})^{\frac{1}{3}})},$$

as claimed. ☐

Similarly, we can deduce that

$$\int_\Omega |X(u^{p_0(1+\delta)})|^2 dx \leq (1+\delta)^2(4M_1)^2 A_0^{2p_0(1+\delta)(1+(\frac{1}{p_0(1+\delta)})^{\frac{1}{3}})}. \tag{2.2.35}$$

Proof of Theorem 2.2.4(1): For $1 < p < \frac{1+\sqrt{1-\varepsilon}}{\varepsilon}$, let $p_1 = p$, there exists a positive integer $k \in \mathbb{N}^+$ such that $(1+\delta)^k \in (1, p]$, and $(1+\delta)^{(k+1)} > p$. Suppose that $p_0 = 1$, and

$$\overline{p_i} = (1+\delta)^i,\ A_i = A_0^{1+\sum_{j=1}^{i}(\frac{1}{1+\delta})^{\frac{j}{3}}},\ \text{for } 1 \leq i \leq k,$$

then for the weak solution $u \in H^1_{X,0}(\Omega)$ with $\|u\|_{L^2(\Omega)} \neq 0$, one has, from the result of Proposition 2.2.11, that

$$\int_\Omega u^{2(1+\delta)^{i+1}}dx = \int_\Omega u^{2\overline{p_i}(1+\delta)}dx \leq A_i^{2\overline{p_i}(1+\delta)\Big(1+(\frac{1}{\overline{p_i}(1+\delta)})^{\frac{1}{3}}\Big)} \leq A_0^{2(1+\delta)^{i+1}(1+\sum_{j=1}^{i+1}(\frac{1}{1+\delta})^{\frac{j}{3}})}.$$

If $\delta = \frac{1}{2M_1} \leq \frac{1}{4}$, then

$$\frac{\log A_k}{\log A_0} = 1 + \sum_{j=1}^{i} \Big(\frac{1}{1+\delta}\Big)^{\frac{j}{3}} \leq 1 + \sum_{j=1}^{\infty} \Big(\frac{1}{1+\delta}\Big)^{\frac{j}{3}} = 1 + 4M_1 \leq 5M_1,$$

where M_1 is independent with i. Thus we have for any $1 \leq i \leq k$,

$$\int_\Omega u^{2(1+\delta)^{i+1}} dx \leq (A_0^{5M_1})^{2(1+\delta)^{i+1}}.$$

Therefore if we choose $A_0 = e^{12}$, $\bar{A} = e^{60M_1}$, $i = k$, then

$$\int_\Omega u^{2(1+\delta)^{k+1}} dx \leq (A_0^{5M_1})^{2(1+\delta)^{k+1}}.$$

This means $u \in L^{2(1+\delta)^{k+1}}(\Omega)$. $(1+\delta)^{k+1} > p$, Ω is bounded, then $u \in L^{2p}(\Omega)$. The result of Theorem 2.2.4(1) is proved. □

Remark 2.2.2. *Observe that if $\varepsilon \to 0+$, then $u \in L^\infty(\Omega)$.*

Lemma 2.2.6. *If $a < 0$, $u_\varepsilon \in C^0(\Omega_1), u_\varepsilon \geq 0, \|u_\varepsilon\|_{L^2(\Omega)} \neq 0$ be a weak solution of* (2.2.19) *on an open set $\Omega_1 \subset \Omega$, then $u_\varepsilon > 0$ for all $x \in \Omega_1$.*

Proof: Suppose that $u_\varepsilon(x_0) = 0$ for some $x_0 \in \Omega_1$, then for any $\lambda > 0$, there exists a small neighborhood $U_0 \subset \Omega_1$ of x_0, such that $0 \leq u_\varepsilon(x) \leq \lambda$ on $\overline{U}_0$. As $a < 0$, we have $au_\varepsilon(x)\log u_\varepsilon(x) + bu_\varepsilon(x) \geq 0$ then $\Delta_X u_\varepsilon \leq 0$ in U_0. But x_0 is a minimum point of u_ε, the Bony's maximum principle implies that $u_\varepsilon \equiv 0$ in U_0. This means that u_ε is a trivial solution from the continuity of u_ε in Ω_1, which is contradiction with the condition $\|u_\varepsilon\|_{L^2(\Omega)} \neq 0$. □

Proof of Theorem 2.2.4(2): Now for $x_0 \in \Omega \setminus \Gamma$, there exist V_0, U_1, U_0 such that $x_0 \in V_0 \subset\subset U_1 \subset\subset U_0 \subset\subset \Omega \setminus \Gamma$, $0 \notin \overline{U}_0$, and for a cut-off function $\phi_0(x) \in C_0^\infty(U_0)$, $\phi_0(x) \equiv 1$ on U_1. let $v_{0,\varepsilon} = \phi_0 u_\varepsilon$, from the equation we know,

$$\begin{cases} -\Delta_X v_{0,\varepsilon} = -u_\varepsilon \Delta_X \phi_0 + \varepsilon V_n \phi_0 u_\varepsilon + a\phi_0 u_\varepsilon \log|u_\varepsilon| + b\phi_0 u_\varepsilon - 2\sum_{j=1}^n X_j\phi_0 X_j u_\varepsilon, & \text{in } U_0, \\ v_{0,\varepsilon} = 0, & \text{on } \partial U_0. \end{cases}$$

Set

$$f_\varepsilon := -u_\varepsilon \Delta_X \phi_0 + \varepsilon V_n \phi_0 u_\varepsilon + a\phi_0 u_\varepsilon \log|u_\varepsilon| + b\phi_0 u_\varepsilon - 2\sum_{j=1}^n X_j\phi_0 X_j u_\varepsilon. \tag{2.2.36}$$

Then for $u_\varepsilon \in L^{2p}(U_0)$, one has for any $1 < \sigma < p$

$$|u_\varepsilon \log|u_\varepsilon||^{\frac{2p}{\sigma}} \leq \Big(\frac{1}{e}\Big)^{\frac{2p}{\sigma}} + C_\sigma |u_\varepsilon|^{2p}, \quad \exists\ C_\sigma > 0. \tag{2.2.37}$$

Hence

$$\int_\Omega |u_\varepsilon \log|u_\varepsilon||^{\frac{2p}{\sigma}} dx \leq \Big(\frac{1}{e}\Big)^{\frac{2p}{\sigma}} |\Omega| + C_\sigma \int_\Omega |u_\varepsilon|^{2p} dx = \Big(\frac{1}{e}\Big)^{\frac{2p}{\sigma}} |\Omega| + C_\sigma \|u_\varepsilon\|^{2p}_{L^{2p}(\Omega)}.$$

So for $\phi_0 \in C_0^\infty(U_0)$, we get

$$\phi_0 u_\varepsilon \log|u_\varepsilon| \in L^{\frac{2p}{\sigma}}(U_0),\ \forall\ 1 < \sigma < p. \tag{2.2.38}$$

On the other hand, $V_n \in C^\infty(\Omega \setminus \{0\})$ and $0 \notin \overline{U}_0$, then

$$\varepsilon V_n \phi_0 u_\varepsilon \in L^{2p}(U_0). \tag{2.2.39}$$

Also from $-\triangle_X u_\varepsilon - \varepsilon V_n u_\varepsilon = a u_\varepsilon \log|u_\varepsilon| + b u_\varepsilon + g(x, u_\varepsilon)$, we have

$$\triangle_X u_\varepsilon \in L^{\frac{2p}{\sigma}}(U_0), \text{ and } X_j^2 u_\varepsilon \in L^{\frac{2p}{\sigma}}(U_0) \text{ for } 1 \le j \le n. \tag{2.2.40}$$

Next, for $1 \le j \le n$, we know

$$\int_\Omega (X_j u_\varepsilon)(X_j u_\varepsilon)^{\frac{2p}{\sigma}-1} dx = -\Big(\frac{2p}{\sigma} - 1\Big) \int_\Omega u_\varepsilon (X_j^2 u_\varepsilon)(X_j u_\varepsilon)^{\frac{2p}{\sigma}-2} dx.$$

Thus we obtain

$$\begin{aligned}\int_\Omega |X_j u_\varepsilon|^{\frac{2p}{\sigma}} dx &\le \Big(\frac{2p}{\sigma} - 1\Big) \int_\Omega |u_\varepsilon X_j^2 u_\varepsilon||X_j u_\varepsilon|^{\frac{2p}{\sigma}-2} dx \\ &\le \Big(\frac{2p}{\sigma} - 1\Big)\Big(\int_\Omega |u_\varepsilon X_j^2 u_\varepsilon|^{\frac{p}{\sigma}} dx\Big)^{\frac{\sigma}{p}} \Big(\int_\Omega |X_j u_\varepsilon|^{\frac{2p}{\sigma}} dx\Big)^{\frac{p-\sigma}{p}},\end{aligned}$$

which means that for $1 \le j \le n$,

$$\int_\Omega |X_j u_\varepsilon|^{\frac{2p}{\sigma}} dx \le \Big(\frac{2p}{\sigma} - 1\Big)^{\frac{p}{\sigma}} \Big(\int_\Omega |u_\varepsilon|^{\frac{2p}{\sigma}} dx\Big)^{\frac{1}{2}} \Big(\int_\Omega |X_j^2 u_\varepsilon|^{\frac{2p}{\sigma}} dx\Big)^{\frac{1}{2}}. \tag{2.2.41}$$

So from $u_\varepsilon \in L^{2p}(U_0)$ and $X_j^2 u_\varepsilon \in L^{\frac{2p}{\sigma}}(U_0)$ $(1 \le j \le n)$, we have, for $\phi_0 \in C_0^\infty(U_0)$,

$$X_j \phi_0 X_j u_\varepsilon \in L^{\frac{2p}{\sigma}}(U_0). \tag{2.2.42}$$

Finally from (2.2.36)-(2.2.42), we gain that

$$f_\varepsilon \in L^{\frac{2p}{\sigma}}(U_0). \tag{2.2.43}$$

Since the system of vector fields X satisfies the finitely type Hörmander's condition on $\Omega \setminus \Gamma$ with Hörmander index Q, then from the results of Proposition , we can deduce that

$$v_{0,\varepsilon} \in M^{2,\frac{2p}{\sigma}}(U_0).$$

Also,

$$u_\varepsilon \in M^{2,\frac{2p}{\sigma}}(U_1), \text{ and } X_j u_\varepsilon \in M^{1,\frac{2p}{\sigma}}(U_1).$$

On the other hand, $1 < \sigma < p$, then for $\varepsilon \in (0, \frac{4}{\bar\nu}(1 - \frac{1}{\bar\nu})) \subset (0, \frac{4}{\sigma\bar\nu}(1 - \frac{1}{\sigma\bar\nu}))$, that implies $2\frac{1+\sqrt{1-\varepsilon}}{\varepsilon} > \sigma\bar\nu$. Therefore for $1 < p < \frac{1+\sqrt{1-\varepsilon}}{\varepsilon}$, we take σ satisfies $\sigma\bar\nu \le 2p$, and then the result of Theorem 1.3.5 (2) implies that

$$v_{0,\varepsilon} \in S^{1,\alpha}(U_0), \ \alpha \in (0, 1 - \frac{\sigma\varepsilon\bar\nu}{2(1 + \sqrt{1-\varepsilon})}).$$

Also,

$$u_\varepsilon \in S^{1,\alpha}(U_1), \alpha \in (0, 1 - \frac{\sigma\varepsilon\bar\nu}{2(1 + \sqrt{1-\varepsilon})}).$$

Then we use the result of Lemma 1.3.1 to get $u_\varepsilon \in C^{\frac{1+\alpha}{Q}}(U_1)$. Also from Lemma 2.2.6, we know $u_\varepsilon(x) \ge \lambda > 0$ for $x \in U_1$, thus

$$u_\varepsilon \log|u_\varepsilon| \in S^{0,\alpha}(U_1), \quad X_j u_\varepsilon \in S^{0,\alpha}(U_1).$$

Similarly we can take U_2 such that $V_0 \subset\subset U_2 \subset\subset U_1$, $\phi_1 \in C_0^\infty(U_1)$, $\phi_1(x) = 1$ on U_2, $v_{1,\varepsilon} = \phi_1 u_\varepsilon$, Then,

$$\begin{cases} -\Delta_X v_{1,\varepsilon} = -u_\varepsilon \Delta_X \phi_1 + \varepsilon V_n \phi_1 u_\varepsilon + a\phi_1 u_\varepsilon \log|u_\varepsilon| + b\phi_1 u_\varepsilon - 2\sum_{j=1}^n X_j\phi_1 X_j u_\varepsilon & \text{in } U_1, \\ v_{1,\varepsilon} = 0 & \text{on } \partial U_1, \end{cases}$$

by using the result of Theorem 1.4.3 and the above estimation, we have finally $v_{1,\varepsilon} \in S^{2,\alpha}(U_1)$, $u_\varepsilon \in S^{2,\alpha}(U_2)$.

For any $k \in \mathbb{N}^+$, we can take $V_0 \subset\subset U_k \subset\subset U_{k-1} \subset\subset \cdots\cdots \subset\subset U_1 \subset\subset U_0$, by the standard iteration procedure, we can prove that $u_\varepsilon \in S^{k,\alpha}(U_k)$, then $u_\varepsilon \in S^{k,\alpha}(V_0)$. This implies, from the result of Lemma 1.3.1, that $u_\varepsilon \in C^{\frac{k+\alpha}{Q}}(V_0)$, i.e. $u \in C^\infty(V_0)$. On the other hand, the result of Lemma 2.2.6 is not hold for the point $x_0 \in \overline{\Omega} \setminus \Gamma$ on the boundary $\partial\Omega$. In this case we can only deduce that $u_\varepsilon \log|u_\varepsilon| \in C^0(V_0 \cap \overline{\Omega})$ even if $u_\varepsilon \in C^{\alpha_1}(V_0 \cap \overline{\Omega})$ for some $\alpha_1 \geq 0$. The result of Theorem 2.2.4(2) is proved. □

2.3 Estimates of Eigenvalues in Infinitely Degenerate Cases

2.3.1 Motivations

We consider the following boundary value problem for infinitely degenerate elliptic equation with a free perturbation,

$$\begin{cases} -\triangle_X u = au\log|u| + bu + f(x), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{2.3.1}$$

where Ω is a bounded open domain of $\mathbb{R}^n$, a, b are constants, $X = \{X_1, X_2, \cdots, X_m\}$ is C^∞ smooth real vector fields defined on Ω, which is infinitely degenerate on a non-characteristic hypersurface $\Gamma \subset \Omega$ and satisfies the finite type of Hörmander's condition with Hörmander index $Q \geq 1$ on $\Omega \backslash \Gamma$. $\triangle_X = \sum_{j=1}^m X_j^2$ is an infinitely degenerate elliptic operator. Here we assume also $\partial\Omega$ is C^∞ smooth and non-characteristic for the system of vector fields X.

Theorem 2.3.1. *If $a > 0$, $f(x) \in L^2(\Omega)$ and X satisfies the logarithmic regularity estimate* (2.1.14) *with $s > 1$. Then the problem* (2.3.1) *has infinitely many nontrivial weak solutions in $H^1_{X,0}(\Omega)$.*

Remark 2.3.1. *In order to prove Theorem 2.3.1, we need the following Perturbation Theorem and estimates of lower bounds of Dirichlet eigenvalues for $-\triangle_X$ (see Proposition 2.3.1 and Theorem 2.3.2 below). For more details of the proof for Theorem 2.3.1, one can refer to [10].*

Proposition 2.3.1 (Perturbation Theorem, c.f. [54])**.** *Suppose $E \in C^1(V)$ satisfies (PS) condition. Let $W \subset V$ be a finite dimensional subspace of V, $w^* \in V \backslash W$, and let $W^* = W \bigoplus \text{span}\,\{w^*\}$; also let*

$$W^*_+ = \{w + tw^*; w \in W, t \geq 0\}$$

denote the upper half-space in W^*. Suppose
(1) *$E(0) = 0$,*
(2) *$\exists R > 0 \ \ \forall u \in W : \|u\| \geq R \Rightarrow E(u) \leq 0$,*

(3) $\exists R^* \geq R \ \ \forall u \in W^* : \|u\| \geq R^* \Rightarrow E(u) \leq 0$,
and let

$$\begin{aligned}\Gamma = &\{h \in C^0(V,V); \quad h \text{ is odd},\ h(u) = u \text{ if } \max\{E(u), E(-u)\} \leq 0, \\ &\quad \text{in particular, if } u \in W \text{ and } \|u\| \geq R, \text{ or if } u \in W^* \text{ and } \|u\| \geq R^*\}.\end{aligned}$$

Then, if

$$\beta^* = \inf_{h\in\Gamma} \sup_{u\in W_+^*} E(h(u)) > \beta = \inf_{h\in\Gamma} \sup_{u\in W} E(h(u)) \geq 0, \tag{2.3.2}$$

the functional E possesses a critical value $\geq \beta^*$.

2.3.2 Lower Bounds of Eigenvalues

Here we consider the eigenvalues of the infinitely degenerate elliptic operator $\triangle_X$ satisfying the logarithmic regularity estimate (2.1.14) with $s > 1$, which implies that $\triangle_X$ is hypoelliptic.

Proposition 2.3.2 (cf. [6, 39])**.** *Suppose that the system of vector fields X satisfies the logarithmic regularity estimate* (2.1.14) *with $s > 1$. If $\partial\Omega$ is C^∞ and non-characteristic for X, then the operator $-\triangle_X$ has a sequence of discrete eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_k \leq \cdots$, and $\lambda_k \to \infty$, such that for any $k \geq 1$, the Dirichlet problem*

$$\begin{cases} -\triangle_X \varphi_k = \lambda_k \varphi_k, & \text{in } \Omega, \\ \varphi_k = 0, & \text{on } \partial\Omega, \end{cases}$$

admits a non trivial solution $\varphi_k \in H^1_{X,0}(\Omega)$. Moreover, $\{\varphi_k\}_{k\geq 1}$ constitute an orthonormal basis of the Sobolev space $H^1_{X,0}(\Omega)$.

Proof: Similar to the proof of Proposition 1.5.3. □

Thus, we have the following result (cf. [7]):

Theorem 2.3.2. *Suppose that the system of vector fields X satisfies the logarithmic regularity estimate* (2.1.14) *with $s > 1$. If $\partial\Omega$ is C^∞ and non-characteristic for X, λ_j is the j^{th} Dirichlet eigenvalue of the problem* (1.5.1)*, then*

$$\sum_{j=1}^{k} \lambda_j \geq C(n,s,\Omega) k (\log k)^{2s} - k, \ \text{ for all } k \geq k_0,$$

where $k_0 = [\frac{2^{2s} e^n B_n |\Omega|_n}{C_0 \pi^n}] + 1$, $C(n,s,\Omega) = (2^n - 1)\Big(C_0 2^{n+4s}(|\log \frac{|\Omega|_n B_n}{(2\pi)^n}|^{2s} + n^{2s})\Big)^{-1}$, B_n is the volume of the unit ball in $\mathbb{R}^n$, $|\Omega|_n$ is the volume of Ω, s and C_0 are given in (2.1.14)*.*

Remark 2.3.2. *If the operator is infinitely degenerate elliptic operator, then the Hörmander index $Q = +\infty$. That means the result in the estimates* (1.5.7) *gives us nothing information for the estimates of the eigenvalues. In this case there is even no any asymptotic results for the eigenvalues estimates. The result of Theorem 2.3.2 is the first result on the lower bound estimates of the Dirichlet eigenvalues for infinitely degenerate elliptic operators.*

Lemma 2.3.1. *For the system of vector fields $X = (X_1, \cdots, X_m)$, if $\{\psi_j\}_{j=1}^k$ are the set of orthonormal eigenfunctions corresponding to the Dirichlet eigenvalues $\{\lambda_j\}_{j=1}^k$. Define*

$$\Psi(x,y) = \sum_{j=1}^{k} \psi_j(x)\psi_j(y).$$

Then for the partial Fourier transformation of $\Psi(x,y)$ in the x-variable,

$$\hat{\Psi}(z,y) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \Psi(x,y) e^{-ix\cdot z} dx,$$

we have

$$\int_\Omega \int_{\mathbb{R}^n} |\hat{\Psi}(z,y)|^2 dz dy = k, \text{ and } \int_\Omega |\hat{\Psi}(z,y)|^2 dy \le (2\pi)^{-n} |\Omega|_n.$$

Proof: Similar to the proof of Lemma 1.5.1. □

Lemma 2.3.2. *Let f be a real-valued function defined on $\mathbb{R}^n$ and $0 \le f \le M_1$. For some $s > 0$, if*

$$\int_{\mathbb{R}^n} f(z) dz \ge 1, \text{ and } \int_{\mathbb{R}^n} (\log(e^2 + |z|^2))^{2s} f(z) dz \le M_2,$$

where $M_2 \ge 2^{4s+n} e^n M_1 B_n$, B_n is the volume of the unit ball in $\mathbb{R}^n$.

Then we have the following inequality,

$$\int_{\mathbb{R}^n} f(z) dz \cdot (\log(\int_{\mathbb{R}^n} f(z) dz))^{2s} \le \frac{2^{n+2s}}{2^n - 1} (|\log(M_1 B_n)|^{2s} + n^{2s}) M_2.$$

Proof of Theorem 2.3.2: First, the problem (1.5.1) has a sequence of discrete eigenvalues $\{\lambda_k\}_{k\ge1}$ and the corresponding eigenfunctions $\{\psi_k(x)\}_{k\ge1}$ constitute an orthonormal basis of the Sobolev space $H^1_{X,0}(\Omega)$.

Taking $\Psi(x,y) = \sum_{j=1}^k \psi_j(x)\psi_j(y)$, then from Lemma 2.3.1, we know

$$\int_\Omega \int_{\mathbb{R}^n} |\hat{\Psi}(z,y)|^2 dz dy = k, \text{ and } \int_\Omega |\hat{\Psi}(z,y)|^2 dy \le (2\pi)^{-n} |\Omega|_n.$$

On the other hand, using Plancherel's formula, we can gain

$$\int_{\mathbb{R}^n} \int_\Omega |\hat{\Psi}(z,y)|^2 (\log(e^2 + |z|^2))^{2s} dy dz = \int_{\mathbb{R}^n} \int_\Omega |(\log(e^2 + |\nabla_x|^2))^s \Psi(x,y)|^2 dy dx.$$

where $\nabla_x = (\partial_{x_1}, \partial_{x_2}, \cdots, \partial_{x_n})$. Next, the logarithmic regularity estimate (2.1.14) gives

$$\begin{aligned}\int_{\mathbb{R}^n} \int_\Omega |(\log(e^2 + |\nabla_x|^2))^s \Psi(x,y)|^2 dy dx \le 2^{2s} C_0 (&\int_\Omega \int_\Omega |X(x)\Psi(x,y)|^2 dx dy \\ &+ \int_\Omega \int_\Omega |\Psi(x,y)|^2 dx dy).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\int_\Omega \int_\Omega |X(x)\Psi(x,y)|^2 dx dy &= \int_\Omega \Big(\sum_{l=1}^m \int_\Omega |\sum_{j=1}^k (X_l(x)\psi_j(x))\psi_j(y)|^2 dx \Big) dy \\ &= \sum_{l=1}^m \Big(\int_\Omega \sum_{j=1}^k |X_l(x)\psi_j(x)|^2 \Big) dx \\ &= -\int_\Omega \sum_{j=1}^k \psi_j(x) \triangle_X \psi_j(x) dx = \sum_{j=1}^k \lambda_j.\end{aligned}$$

Therefore, from the above calculations, we obtain

$$\int_{\mathbb{R}^n}\int_{\Omega}|\hat{\Psi}(z,y)|^2(\log(e^2+|z|^2))^{2s}dydz \le 2^{2s}C_0(\sum_{j=1}^{k}\lambda_j+k).$$

Now we choose

$$f(z)=\int_{\Omega}|\hat{\Psi}(z,y)|^2dy,\ M_1=(2\pi)^{-n}|\Omega|_n,\ M_2=2^{2s}C_0(\sum_{j=1}^{k}\lambda_j+k).$$

Then we know that $0\le f(z)\le M_1$, if we take $k_0=[\frac{2^{2s}e^nB_n|\Omega|_n}{C_0\pi^n}]+1$, then for any $k\ge k_0$, we can see

$$\int_{\mathbb{R}^n}f(z)dz=k\ge 1,\ \text{and}\ M_2\ge 2^{4s}e^n|\Omega|_nB_n\pi^{-n}=2^{4s+n}e^nM_1B_n.$$

Thus from the result of Lemma 2.3.2, for any $k\ge k_0$, we have

$$k(\log k)^{2s}\le \frac{2^{n+4s}}{2^n-1}(|\log\frac{|\Omega|_nB_n}{(2\pi)^n}|^{2s}+n^{2s})C_0\cdot(\sum_{j=1}^{k}\lambda_j+k).$$

That means, for any $k\ge k_0$,

$$\sum_{j=1}^{k}\lambda_j\ge C_3k(\log k)^{2s}-k,$$

where $C_3=(2^n-1)\Big(C_02^{n+4s}(|\log\frac{|\Omega|_nB_n}{(2\pi)^n}|^{2s}+n^{2s})\Big)^{-1}$. □

Proof of Lemma 2.3.2: We choose a constant $R>0$ such that

$$\int_{\mathbb{R}^n}(\log(e^2+|z|^2))^{2s}g(z)dz=M_2,$$

where

$$g(z)=\begin{cases}M_1, & |z|<R,\\ 0, & |z|\ge R.\end{cases}$$

Since $M_2\ge 2^{4s+n}e^nM_1B_n$, that means $R\ge 2e$. In fact, if $R<2e$, then

$$\begin{aligned}M_2&=\int_{\mathbb{R}^n}(\log(e^2+|z|^2))^{2s}g(z)dz=M_1\omega_{n-1}\int_0^R(\log(e^2+r^2))^{2s}r^{n-1}dr\\ &\le M_1B_n(\log(5e^2))^{2s}(2e)^n<2^{4s+n}e^nM_1B_n,\end{aligned}$$

where ω_{n-1} is the area of the unit sphere in $\mathbb{R}^n$, B_n is the volume of the unit ball in $\mathbb{R}^n$ and $nB_n=\omega_{n-1}$. Which is incompatible with the condition of M_2.

By $R\ge 2e$, one has $R\ge 2\sqrt{R}$, and

$$\begin{aligned}M_2&\ge M_1\omega_{n-1}\int_{\frac{R}{2}}^{R}(\log(e^2+r^2))^{2s}r^{n-1}dr\ge M_1\omega_{n-1}2^{2s}\int_{\frac{R}{2}}^{R}(\log r)^{2s}r^{n-1}dr\\ &\ge 2^{2s}M_1B_n(1-2^{-n})R^n\Big(\log\frac{R}{2}\Big)^{2s}\ge M_1B_n(1-2^{-n})R^n(\log R)^{2s}.\end{aligned}\tag{2.3.3}$$

Since $\Big[(\log(e^2+|z|^2))^{2s}-(\log(e^2+R^2))^{2s}\Big](f(z)-g(z))\geq 0$, we have

$$(\log(e^2+R^2))^{2s}\int_{\mathbb{R}^n}\big(f(z)-g(z)\big)dz\leq\int_{\mathbb{R}^n}(\log(e^2+|z|^2))^{2s}\big(f(z)-g(z)\big)dz\leq 0,$$

which implies

$$\int_{\mathbb{R}^n}f(z)dz\leq\int_{\mathbb{R}^n}g(z)dz. \tag{2.3.4}$$

Using (2.3.4) and the fact $\int_{\mathbb{R}^n}f(z)dz\geq 1$, we can obtain

$$\begin{aligned}\int_{\mathbb{R}^n}f(z)dz\cdot\Big(\log(\int_{\mathbb{R}^n}f(z)dz)\Big)^{2s}&\leq\int_{\mathbb{R}^n}g(z)dz\cdot\Big(\log(\int_{\mathbb{R}^n}g(z)dz)\Big)^{2s}\\&=M_1B_nR^n\cdot\Big[\log(M_1B_nR^n)\Big]^{2s}\\&\leq M_1B_nR^n\cdot 2^{2s}(|\log(M_1B_n)|^{2s}+(n\log R)^{2s})\\&\leq 2^{2s}M_1B_n(|\log(M_1B_n)|^{2s}+n^{2s})R^n(\log R)^{2s}.\end{aligned}\tag{2.3.5}$$

From the estimates (2.3.3) and (2.3.5), we can deduce that

$$\int_{\mathbb{R}^n}f(z)dz\cdot\Big(\log(\int_{\mathbb{R}^n}f(z)dz)\Big)^{2s}\leq\frac{2^{n+2s}}{2^n-1}(|\log(M_1B_n)|^{2s}+n^{2s})M_2.$$

□

2.3.3 Summary: Finitely Degenerate Elliptic Operators and Infinitely Degenerate Elliptic Operators

Finally, let us compare the results between finitely degenerate vector fields and infinitely degenerate vector fields.

If the system of vector fields X satisfies the Hörmander's condition, then $\triangle_X$ is the finitely degenerate elliptic operator, and the following conditions are equivalent:

(1) The vector fields X is a finitely degenerate with Hörmander index Q.

(2) Sub-elliptic estimate:

$$\big\||\nabla|^{\frac{1}{Q}}u\big\|^2_{L^2(\Omega)}\leq C_1\|Xu\|^2_{L^2(\Omega)}+C_2\|u\|^2_{L^2(\Omega)},$$

holds for all $u\in C_0^\infty(\Omega)$, and some $C_1>0$ and $C_2\geq 0$.

(3) There exists $C>0$, such that for $x\in\Omega$, $r>0$, we have $B_E(x,r)\subset B_X(x,Cr^{\frac{1}{Q}})$, where B_E is the Euclid ball and B_X is the sub-elliptic ball induced by sub-elliptic metric (which is also C-C metric).

Remark 2.3.3. (1) *Sub-elliptic estimates imply the hypoellipticity of* $\triangle_X$.

(2) *From the condition (3) above, doubling property holds for* B_X*. Thus Sobolev inequality and Poincaré inequality are all hold.*

If the system of vector fields X is an infinitely degenerate vector fields and satisfies the following logarithmic regularity estimate

$$\|(\log\Lambda)^s u\|^2_{L^2(\Omega)}\leq C\big(\|Xu\|^2_{L^2(\Omega)}+\|u\|^2_{L^2(\Omega)}\big),\ \text{for any } u\in C_0^\infty(\Omega), \tag{2.3.6}$$

with $s > 1$, where $\Lambda = (e^2 + |\nabla|^2)^{1/2}$. Then we have

(1) The infinitely degenerate elliptic operator $\triangle_X$ is hypoelliptic.
(2) The C-C distance induced by X can be defined which might be a non-doubling metric.

Remark 2.3.4. (1) *If the vector fields X is an infinitely degenerate vector fields, then the sub-elliptic estimates will be not satisfied. Thus, the regularity of the infinitely degenerate elliptic operator $\triangle_X$ can be deduced by the logarithmic regularity estimate for X.*

(2) *If X is an infinitely degenerate vector fields, the Sobolev inequality will be not satisfied. However, in this case we have the following logarithmic Sobolev inequality:*

Suppose that the vector fields $X = (X_1, \cdots, X_m)$ satisfies the logarithmic regularity estimate (2.3.6) *for $s > \frac{1}{2}$. Then*

$$\int_\Omega |u|^2 |\log(\frac{|u|}{\|u\|_{L^2(\Omega)}})|^{2s-1} dx \le C_0 \Big[\int_\Omega |Xu|^2 dx + \|u\|^2_{L^2(\Omega)}\Big], \text{ for all } u \in H^1_{X,0}(\Omega).$$

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